LĪLĀVATĪ

$(65)^2 = (x+y)^2$

of

Bhāskarācārya

Part I

$= 6^2 + 2.6.5 + 5^2 = x^2 + 2xy + y^2$

$= 36 = x^2$

$60 = 2xy$

$25 = y^2$

$4225 = x^2 + 2xy + y^2$

by

M. D. PANDIT

LĪLĀVATĪ

of

Bhāskarācārya

(Translated into English with Notes and Appendices)

Part I
(parikarmāṣṭaka)

by
M. D. PANDIT

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To,

Sarasvatī

The Goddess of Learning

FOREWORD

It is with great pleasure that we are placing this Selection (on a new and hitherto unprescribed subject) in the hands of the M. A. students of Sanskrit in the University of Poona. The Sanskrit syllabi of the various courses in the Universities of our country are found to be rotating round kāvya, philosophy, religion and a few other comparatively minor subjects. An outsider may think, and he would be justified in doing so, that the Sanskrit literature does not go beyond these subjects and that this ancient land of ours did not give any thought to scientific and technical subjects. The University Grants Commission therefore appointed a Committee of senior professors drawn from various Universities in India and directed it to go into the question of reviewing and suitably revising the Sanskrit curriculum. The committee's meetings were held in the University of Poona and in August 1989 it brought out a voluminous report containing many useful suggestions for improving the Sanskrit studies in our Universities. One of the suggestions was that a paper on scientific and technical literature in Sanskrit be introduced at the M.A. Level. The Board of Studies in Sanskrit, Pali and Ardhamagadhi of our University accepted this suggestion and resolved to prescribe portion of texts on subjects like Mathematics, Engineering, Medicine, Chemistry, Agriculture, Gajavidyā, Aśvavidyā etc. The present selection is on the subject of Mathematics.

When the question of prescribing a text on 'Gaṇita' came up, the first and foremost name suggested by the members was that of the Līlāvatī of Bhāskarācārya. It was therefore decided to select and appoint portions from the Līlāvatī. As it is a text for a section of paper II of M. A. Subordinate level, it was necessary to restrict our choice to the basic functions in Gaṇita so that even those students who do not offer Sanskrit

at the Principal level, should be able to follow the subject without much effort, the main aim being just to aquaint the students with the preliminaries of this science. Portion pertaining to addition, subtraction, multiplication and division is taken in this selection. This is a basic part of Arithmetic and the students will not find any difficulty in following it though it is in Sanskirt.

Bhāskarācārya is the brightest star in the firmament of Indian Mathematics. He is Bhāskarācārya II as there was another mathematician of the same name who flourished in the 7th century. But our Bhāskarācārya outshone him and all others who lived before him. He was a Maharashtrian Brahmin born in the year 1114 A. D. and lived at Vijjalaviḍa, a town which cannot be indisputably identified today, but in all probability a village or town situated between Jalgaon and Chalisagaon. In the Golādhyāya of his great work 'Siddhānta-śiromaṇi' he gives information about his genealogy starting from Trivikrama. He belonged to the Śāndilya Gotra. Every ancestor of our author was learned and well-versed in the various sāstras. His father, Maheśwara, was known for his great learning in the Śrauta and Smārta sciences and specially in Mathematics and Astronomy. Bhāskarācārya learnt all these Sāstras at the feet of this great Guru, his own father. In addition to being a great 'Gaṇit' Bhāskarācārya was a Kavi which is a fact. He has written all his works in verse and his writings are not only mathematically accurate, but are also poetically charming. As he himself has mentioned in the last verse of the Līlāvatī, he was in addition, well-versed in the eight systems of grammar, six systems of medicine, six of Logic, five of mathematics., the four Vedas and the two Mīmāṁsās (the sacriticial science and the Vedānta) But in the end his mind was fixed on that One Brahman which is the Highest Reality in this universe. He left this world in 1193.

He wrote several works the chief among them being the 'Siddhānta-Śiromaṇi' which he wrote in 1150 A. D. at the age of 36. This magnum opus consists of four parts - Līlāvatī, Bījagaṇita, Gaṇitādhyāya and Golādhyāya. In addition, he wrote 'Karaṇa-Kutūhala', 'Sarvatobhadra-yantra' 'Vivāha-paṭala', 'Vasiṣṭhatulya' etc. Bhāskarācārya was the pride of India. He foreshadowed many theories which were discovered and developed by western thinkers two or three centuries later.

The Līlāvatī is the first prakaraṇa in his Siddhānta Siromani. It was so popular that the great emperor Akbar got it translated into Persian by Abul Fazel. The second prakaraṇa - Bījagaṇita

also was translated into Persian by Atta Ulla Rashdee. He calls it 'Pāṭī-gaṇita' i.e. vyakta-gaṇita i.e. Arithmatic though it contains some topics from geometry and algebra. It contains 261 (or 277 or 288) verses. All are simple and beautiful. It became so popular that as many as 24 commentaries by very learned persons have been written on it. There are many stories about the name Līlāvatī. Some say that Bhāskarācārya wrote it for his daughter 'Līlā' who was his pupil. Others say that she became a widow at an early age and to amuse her and to keep her mind engaged her father wrote it. Still others say that the name refers to the easy and flowing style of the book. It is hard to believe any of these except the first one to some extent.

When the question of text-book on Līlāvatī came up, the only way was found to be to write an widependent, new text for the students. The only person, who was whanimously thought to be well qualified for the job of translating Līlāvati was Dr. M. D. Pandit. And fortunately enough, Dr. Pandit also kindly agreed to do the job.

Dr. M. D. Pandit of the Department of Sanskrit and Prakrit Languages, in our University who is an expert in the fields of Veda, Vedānta, Vyākaraṇa, Sanskrit Mathematies, Astronomy, Astrology, as well as the modern science of Linguisties and Comparative Philology, and who has a number of research-publications of the highest order to his credit, has accordingly prepared an excellent edition of the first 30 verses of the Līlāvatī. It contains all the necessary material on this subject and the students will derive great benefit from its use both in understanding the subject and in preparing for the examination.

Ahmednagar,
August 1, 1992.

Prof. N. N. Kolhapure,
Chairman,
Board of Studies in Sanskrit,
Pali & Ardhamagadhi,
University of Poona.

Preface

1. Introductory : If we try to trace the history of mathematics in India, we find that we have to go as far back as the Vedic times. Though mathematics is not the subject or scope of the Vedas, they do contain data relating to mathematics. If we search for such mathematical data, we find that the number-system as enunciated by the Vedas is not different from the one which we are following throughout the world today. Actually, the truth is that we have based all our present mathematics on the number-system of the Vedas and that we have not contributed anything to and not improved on the Vedic number-system.

The chief characteristic of the Vedic number-system is that it takes the number 'ten', *daśa* (written in symobols as 10) as the base or verandah for changing over to the next rank or level or series. In other words, the Vedic number-system is a decimal number-system, dividing the number-ranks on the basis of 10 and its multiples. Another characteristic of the Vedic number-system is the value of the numbers in terms of place or what in modern mathematics is called as the rank or level. Thus, from numbers 1 to 9, we have all single-digits, but from 10 onwards upto 99, we have two-digits; from 100 to 999 we have three digits and so on. Though in numbers 99 or 999 or 9999 all the digits are identical viz. 9, their value differs according as they are placed to the right or the left in the number. Thus, the left 9 in 99 is ten times greater in value than the right 9; the left 9 in 999 is hundred times and ten times greater in value than the second left 9 and the right 9 respectively. The place-value notation thus proved to be of great advantage in developing the number-system.

One of the most important result of the place-value notation has been that it gave rise to one of the greatest invention in mathematics, viz. Zero, which revolutionised the mathematics of the world. Thus, when there are two places and the digit is only one, the other place which remained vacant after writing the one digit was indicated by zero, symbolised as O. Thus, if there are

the two places of *ekam* and *daśa,* and only the number for *daśa* is given, the *ekam* place was symbolised by zero; to illustrate, if the number 9 is given as occupying the *daśa* place and no number for *ekam* place is given, the whole number was written as -

daśa	ekam
9	0

Because of this device of indicating the vacant place or rank or level by zero, the number-system with its poteti̇ality increased infinitely could be theoretically expanded upto any infinite limit. Thus, we can go on expanding the number-system starting from 1 as 1, 2... 10... 100... 1000... 100000000 to infinity. And it must be remembered that all this has been inherited by not only Indians but the whole humanity from the Vedas which are the oldest literature of the world. Even the Vedas refer to still older scientists from whom they have borrowed this knowledge. The number-system in the Vedas is also a fully-developed number-system. This will show how much ancient the tradition of Indian mathematics[1] is !

History[2] tells us that the Babylonian mathematics had a scale of sixty; also that the Mayan mathematics was based on twenty. Yet, in spite of the spread of these ancient scales, the Vedic number - system based on ten surpassed them all - so much so that like the Pāṇinian grammar, it not only surpassed all these mathematical systems but threw them, first, into back-ground and then into total oblivion; the number-systems, other than the decimal, have remained only in history and have gone totally out of use in mathematics. Even the scales of Twenty and Sixty came to be transformed in terms of the scales of Ten. This state of affairs speaks for the convenience, ease and the most natural suitablity of the Vedic number

1. for details, of M. D. PANDIT, *Mathematics As known To the Vedas,* Indian Books Centre, New Delhi, 1992.

2. cf D. E. SMITH, *History of Mathematics,* Dover Publications, New York, 1958, Vol. I, PP. 35-52.

system based on the basis of ten and on the place-value notation. The reason might be that the base ten seems to be the most natural one and suitable to the present structure of the universe.

2. DEVRAO UKHĀ SHET YERAṆḌOLKAR who translated for the first time *Līlāvati* has quoted some verses on the traditions of some *ṛṣis* who were the mathematicians in ancient India. These verses, he says in his *Prastāvanā* (= Introduction to his translation), were sent to him by his friend, Shri VĀMAN VYAṄKEṬEŚA EKBOTE, Saṅgamnerkar, Nimtāndar, Nisbat Gaekwad Sarkar. EKBOTE, it is said there, collected these verses from different books which are unfortunately not mentioned in the *Prastāvanā*. The source of these verses is, therefore, totally unknown. The verses are reproduced here only in order to bring to light the fact that although we have very scanty information about the mathematical activity in ancient India, there had been a very extensive study of mathematics and that the ancient Indian mathematicians were not ignorant of the status of mathematics as a pure science. Even the study of mathematics had different and long traditions. The verses are as follows. Mathematics, as will be clear from the following verses, was called in ancient times as the गणिर्वेद :-

विधाताऽथर्वसर्वस्य गणिर्वेदं प्रकाशयन् ।
स्वनाम्ना संहितां चक्रे लक्षश्लोकमयीमृजुम् ॥ 1 ॥

ततः प्रजापतिं दक्षं दक्षं सकलकर्मसु ।
विधिर्धीनीरधिं सांगं गणिर्वेदमुपादिशत् ॥ 2 ॥

एकदा हिमवत्पार्श्वे दैवादागत्य संगतः (ताः ?)।
मुनयो बहवस्तेषां नामानि कथयाम्यहम् ॥ 3 ॥

भरद्वाजो मुनिवरः प्रथमं समुपागतः ।
ततोंऽगिरास्ततो गर्गो मरीचिर्भृगुभार्गवौ ॥ 4 ॥

पुलस्त्योऽगस्तिरसितो वसिष्ठश्च पराशरः ।
हारितो गौतमः सांख्यो मैत्रेयश्च्यवनोऽपि च ॥ 5 ॥

जमदग्निश्च गार्ग्यश्च कश्यपः काश्यपोऽपि वा ।
नारदो वामदेवश्च मार्कण्डेयः कपिंजलः ॥ 6 ॥

शांडिल्यः सहकौण्डण्यः शाकुनेयश्च शौनकः ।
आश्वलायनसांकृत्यौ विश्वामित्रः परीक्षकः ॥ 7 ॥

देवलो गालवो धौम्यः काम्यकात्यायना उभौ ।
कांकायनो वैजवापः कुशिको बादरायणिः ॥ 8 ॥

हिरण्याक्षश्च लौगाक्षिः शरलोमा च गोभिलः ।
वैखानसा वालखिल्यस्तथैवान्ये महर्षयः ॥ 9 ॥

इत्थं स मुनिभिर्योग्यैः प्रार्थितो विनयान्वितैः ।
भरद्वाजो मुनिश्रेष्ठो जगाम त्रिदशालयम् ॥ 10 ॥

तत्रेन्द्रभवनं गत्वा सुरर्षिगणमध्यगम् ।
दृष्टवान् वृत्रहन्तारं दीप्यमानमिवानलम् ॥ 11 ॥

दृष्ट्वैव स मुनिं प्राह भगवान् मघवा मुदा ।
धर्मज्ञ स्वागतं तेऽद्य मुनिं तं समपूजयत् ॥ 12 ॥

सोऽभिगम्य जयाशीर्भिरभिनन्द्य सुरेश्वरम् ।
ऋषीणां वचनं सम्यक् श्रावयन् मुनिसत्तमः ॥ 13 ॥

मूढो यो हि समुत्पन्नः सर्वप्राणी न बुद्धिमान् ।
तेषां प्रशमनोपायं यथावद् वक्तुमर्हसि ॥ 14 ॥

तन्त्रस्य कर्ता प्रथममग्निवेशोऽ भवत् पुरा ।
ततो भेडादयश्चक्रुः स्वस्वतन्त्राणि तानि च ॥ 15 ॥

श्रावयामासुरात्रेयं मुनिवृन्देन वन्दितम् ।
श्रुत्वा तानि (च) तन्त्राणि हृष्टोऽभूदभिनन्दनः ॥ 16 ॥

यथावत् सूत्रितं तस्मात् प्रहृष्टा मुनयोऽभवन् ।
दिवि देवर्षयो देवाः श्रुत्वा साध्विति तेऽब्रुवन् ॥ 17 ॥

आगमश्चन्द्रसेनश्च लंकेशश्च विशारदः ।
कपालिमतमाण्डव्यौ भास्करः सुरसेनकः ॥ 18 ॥

रत्नकोपश्च शंभुश्च तथैको नरवाहनः ।
इन्द्रदो गोमुखश्चैव कंबलिर्व्यालिरेव च ॥ 19 ॥

नागार्जुनः सुरानन्दो नागबोधिर्यशोधनः ।
खंडः कपालिको ब्रह्मा गोविंदो लंपको हरिः ॥ 20 ॥

गण्यांकुशो भैरवश्च काकचण्डीश्वरस्तथा ।
वासुदेव ऋष्यशृंगः क्रियातन्त्रसमुच्चयी ॥ 21 ॥

गण्येन्द्रतिलको योगी भाह्लुकी मैथिलाह्वय: ।
महादेवो महेंद्रश्च रत्नाकरो हरीश्वर: ॥ 22 ॥

एते चान्ये च कौत्सश्च सिद्धा: शास्त्रप्रवर्तका: ।
विरोचनौ (ना + उ) वाच ॥ 23 ॥

प्रवासिन: कालसस्य गणिर्विद्यां प्रकाशयेत् ।
कल्याणाख्येन सुज्ञेन त्वच्छिष्येण च दीयते ॥ 24 ॥

तत्क्षणात् कोपसंविष्ट: ऋषि: शिष्यस्य शापयेत् ।
भस्मीभूतश्च कल्याणो विद्या या निष्फला भवेत् ॥ 25 ॥

We know from the last verse that one कल्याण, a disciple of a mathematician sage, imparted the knowledge of गणिर्विद्या to a foreigner - traveller called कलस; the enranged sage then cursed कल्याण and reduced him to ashes. This कलस then seems to have taken the knowledge of mathematics to other countries. The whole history of ancient Indian mathematics requires to be studied very deeply.

3. In spite of the advanced stage of development and the highest degree of the convenience and ease of notation, the mathematical system in India seems to have been neglected as a theory. We do not get even a single book as a text on pure mathematics of real/complex numbers before the times of Bhāskarācārya. Bhāskarācārya's *Līlāvatī* is the first known text on pure mathematics of real numbers. Why ?

The reason seems to be that we find that mathematics in ancient India, even in Vedic times, has always served as the hand-maid of other sciences like Astronony and Astrology. It never seems to have been separated from its applicational nature. All the books on Astronomy right from the first astronomical treatise of *Vedāṅga Jyotiṣa* to those by Āryabhat I and II and Varāhamihira, employ mathematics to their purposes and do not seem to care for its status as a pure and theoretical science. The same state of affairs exists even in the times after Bhāskarācārya. In a sense, Bhāskarācā*rya's Līlāvatī* seems to be the only perfect text-book on mathematics, explaining and stating the rules and methods of the different mathematical operations. The Vedas, and all the

consequent Sanskrit literature up to the times of Bhāskarācārya have applied mathematics in other sciences; but none of them seems to have attempted its development as a pure science. In this sense, the importance of Bhāskarācārya's *Līlāvatī* can hardly be exaggerated.

4. *Bhāskarācārya* :- Bhāskarācārya was a mathematician as well as a learned astronomer. He has written two books which have been held as authoritative even to-day. In astronomical matters, he improved on Brahmagupta as well as Āryabhaṭa. The titles of the two books are : सिद्धान्तशिरोमणि and करणकुतूहल.

Bhāskarācārya seems to have been born in Śaka 1036 (= 1114 AD) and he wrote his सिद्धान्तशिरोमणि in Śaka 1072 (= 1150 AD). This is clear from his narrative of his own self in verse 58 in the chapter on गोलाध्याय in सिद्धान्तशिरोमणि. The verse runs as follows :-
रसगुणपूर्णमही (= 1036) समशकनृपसमयेऽभवन् ममोत्पत्तिः । रसगुण (= 36) वर्षे मया सिद्धान्तशिरोमणी रचितः ॥

He has also written his own commentary called वासनाभाष्य on the two अध्यायs of ग्रहगणित and गोल in his सिद्धान्ताशिरोमणि. He wrote the other book called करणकुतूहल in *Śaka* 1105 (= 1183 AD) as his remark in the वासनाभाष्य, viz. तथा शरखण्डकानि मया करणे कथितानि shows; also his acceptance of 11 अयनांश points to the same date for the composition of करणकुतूहल viz. *śaka* 1105. This means he composed करणकुतूहल at his age of 69.

From verses nos. 61 and 62 in the गोलाध्याय, we get some information about his family. His father's name was महेश्वर and his गोत्र was शाण्डिल्य. He resided in a town called विज्जडविड or विज्ञलबिड near the ranges of the Sahya-mountains. After all pros and cons of the above statement, S. B. DĪKṢITA (cf भारतीय ज्योतिषशास्त्र, P. 247-248) comes to the conclusion that विज्ञलविड seems to be the modern पाटण near the village चांदवड on the border of the Nashik and Aurangabad districts. N. H. PHADKE (of. लीलावती पुनर्दर्शन, प्रस्तावना, PP. 16-17) does not accept the above argument. He, however, expresses total inability to locate the town विज्ञलविड. The verses nos. 61 and 62 are as follows :-

आसीत् सह्यकुलाचलाश्रितपुरे त्रैविद्यविद्वज्जने
नानासज्जनधाम्नि विज्जलविडे शांडिल्यगोत्रो द्विज: ।
श्रौतस्मार्तविचारसारचतुरो नि:शेषविद्यानिधि:
साधूनामवधिर्महेश्वरकृती दैवज्ञचूडामणि: ॥ 61 ॥

तज्जस्तच्चरणारविन्दयुगलप्राप्तप्रसाद: सुधी:
मुग्धोद्बोधकरं विदग्धगणकप्रीतिपदं प्रस्फुटम् ।
एतद् व्यक्तसदुक्तियुक्तिबहुलं हेलावगम्यं विदाम्
सिद्धान्तप्रथनं कुबुद्धिमथनं चक्रे कविर्भास्कर: ॥ 62 ॥

The inscription by चंगदेव, who was a prominent astrologer in the court of the king सिंघण of देवगिरी (from *śaka* 1132 to *śaka* 1159) and who was the grand-son of भास्कराचार्य throws some more light on भास्कराचार्य's family[3]-tree. The verses are as follows[4] :-

शाण्डिल्यवंशे कविचक्रवर्ती
त्रिविक्रमोऽभूत् तनयोऽस्य जात: ।
यो भोजराजेन कृताभिधानो
विद्यापतिर्भास्करभट्टनामा ॥ 17 ॥

तस्माद् गोविंदसर्वज्ञो जातो गोविंदसन्निभ: ।
प्रभाकर: सुतस्तस्मात् प्रभाकर इवापर: ॥ 18 ॥

तस्मान्मनोरथो जात: सतां पूर्णमनोरथ: ।
श्रीमान् महेश्वराचार्यस्ततोऽजनि कवीश्वर: ॥ 19 ॥

तत्सूनु: कविवृन्दवन्दितपद: सद्वेदविद्यालता-
कन्द: कंसरिपुप्रसादितपद: सर्वज्ञविद्यासद: ।
यच्छिष्यै: सह कोऽपि नो विवदितुं दक्षो विवादी क्वचित्
श्रीमान् भास्करकोविद: समभवत् सत्कीर्तिपुण्यान्वित: ॥ 20 ॥

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3. The inscription, deciphered and discovered by *Bhāu Dājī,* was first printed in *JRAS,* Vol. I, P. 414; it was again reprinted in *Epigraphic Indica,* Vol. I, P. 340, It mentions the name of the town पाटण.

4. N. H. PHADKE *(ibid.* p. 200) has quoted the full text of the inscription. Only relevant portion is quoted here. He has also reproduced a दानपत्र to चांगदेव in Ahirāṇi language (p. 201)

लक्ष्मीधराख्योऽखिलसूरिमुख्यो
वेदार्थवित् तार्किकचक्रवर्ती ।
क्रतुक्रियाकाण्डविचारसार-
विशारदो भास्करनन्दनोऽभूत् ॥ 21 ॥

सर्वशास्त्रार्थदक्षोऽयमिति मत्वा पुरादतः ।
जैत्रपालेन यो नीतः कृतश्च विबुधाग्रणीः ॥ 22 ॥

तस्मात् सुतः सिंघणचक्रवर्ति-
दैवज्ञवर्योऽजनि चङ्गदेवः ।
श्रीभास्कराचार्यनिबद्धशास्त्र -
विस्तारहेतोः कुरुते मठं यः ॥ 23 ॥

भास्कररचितग्रन्थाः सिद्धान्तशिरोमणिप्रमुखाः ।
तद्वंश्यकृताश्चान्ये व्याख्येया मन्मठे नियमात् ॥ २४ ॥

The above 8 verses give us the following family-tree of भास्कराचार्य :-

त्रिविक्रम
।
भास्करभट्ट
।
गोविन्द
।
प्रभाकर
।
मनोरथ
।
महेश्वर
।
भास्कर
।
लक्ष्मीधर
।
चंगदेव,

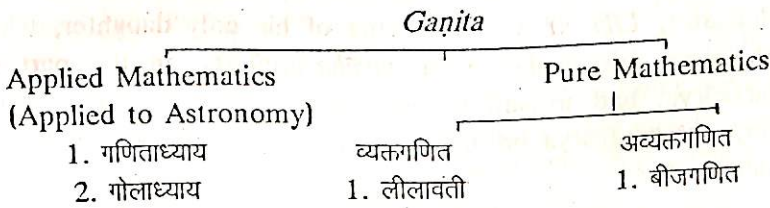
out of which the 3rd name भास्कर from below refers to the present भास्कराचार्य, the author of लीलावती. Though लक्ष्मीधर, 2nd from below,

was invited by king जैत्रपाल to serve in his court, and also चंगदेव was the royal astronomer and astrologer in the court of the king सिंघण, भास्कराचार्य is nowhere mentioned as belonging to the court of any king.

Besides the two books, Bhāskarācārya is also said to have written two more books, one entitled भास्करव्यवहार and the other called (भास्करकृत) विवाहपटल. But the authorship of these two latter books is doubtful.

5. *Līlāvatī* : The book called *Līlāvatī* forms the first Adhyāya or Khaṇḍa of Bhāskarācārya's big work entitled सिद्धान्तशिरोमणि. The work सिद्धान्तशिरोमणी is divided into four main Adhyāyas or Khaṇḍas which are again divided into smaller sub-Adhyāyas or sub-Khaṇḍas. They are (i) *Līlāvatī,* (ii) *Bījagaṇita,* (iii) *Gaṇitādhyāya* and lastly (iv) *Golādhyāya.* While the chapters (iii) and (iv) are applied mathematics, as applied in astronomy, the chapters (i) and (ii), viz. *Līlāvatī* and *Bījagaṇita* are pure mathematics of known and unknown numbers. Again, Bhāskarācārya seems to divide mathematics into two groups, viz. व्यक्तगणित and अव्यक्त गणित. Of these, *Līlāvatī* forms the part of व्यक्तगणित and *Bījagaṇita* forms the part of अव्यक्तगणित. Thus to represent the contents of सिद्धान्तशिरोमणि in the form of a diagramme,

Gaṇita

Applied Mathematics (Applied to Astronomy)	Pure Mathematics	
	व्यक्तगणित	अव्यक्तगणित
1. गणिताध्याय	1. लीलावंती	1. बीजगणित
2. गोलाध्याय		

व्यक्तगणित is also called as पाटीगणित or अंकगणित; अव्यक्तगणित is called as बीजगणित. Thus we can see that the structure of सिद्धान्तशिरोमणि is based on a definite pattern.

Though *Līlāvatī* and *Bījagaṇita* form part of a bigger field of mathematics and of a single book called सिद्धान्तशिरोमणि, they are written in such a way and style that they are self-sufficient and donot require the help of other parts of the book. They thus create an impression that they are independent works. Such is not the case

with the other two parts viz. गणिताध्याय and गोलाध्याय.

The contents of *Līlāvatī* relate to two main topics in mathematics, viz. अंकगणित, dealing with the real numbers and their operational relation with one another and महत्त्वमापन, concerning with the finding out of areas and volumes of things. The whole text is prefixed with different tables of measurements of weights, volumes and length.

Līlāvatī contains in all 261 verses according to N. H. PHADKE. YERANDOLKAR'S edition shows the total number of verses as 280 and S. B. DĪKṢITA mentions the total number of verses as 278. It is difficult to decide the exact number. A critical edition of *Līlāvatī* is a desideratum.

6. The title *Līlāvatī* :- There are a lot of legends about Bhāskarācārya and about the title of his text-book on पाटीगणित, viz. *Līlāvatī*. If the number of legends about a person is any criterion of his fame and name and popularity, Bhāskarācārya can safely be counted as the most famous and popular personality of his times, and even of later times, since no other personality in the history of Sanskrit literature has the fortune of getting so many legends.

According to one tradition or legend, which is current only in Mahārāstra, *Līlāvatī* was the name of his only daughter, who, through some miscalculation of *vivāha-muhūrta* on the part of Bhāskarācārya, had to suffer widowhood. After this unfortunate incident, Bhāskarācārya brought her back to his house and taught her mathematics.

According to the second version of the story, *Līlāvatī* was his daughter; but sensing, on the basis of the calculation of the planet-positions in her horoscope, that she would be widow, Bhāskarācārya cancelled her marriage altogether. *Līlāvatī* then remained unmarried for the whole life. Bhāskarācārya then taught her mathematics so that she could pass her leisure time.

But taking into consideration the times in which Bhāskarācārya and Līlāvati lived and in which learning of any kind - much less शास्त्रविद्या was denied to ladies (and Līlāvatī was a widow) it is

not possible to believe the first legend.

As regards the second one, it is impossible to imagine that any wise father would keep his daughter unmarried for fear of her widowhood.

The third legend, which is current in the North India, pictures Līlāvatī as Bhāskarācārya's wife who did not beget any son for him. Bhāskarācārya, therefore, taught her mathematics to enable her to spend her pastime.

This seems to be totally false because, on the basis of the inscription quoted before, Bhāskarācārya had a son name *Lakṣmīdhara.*

The fourth legend narrates that Līlāvatī was the daughter of a teacher under whom Bhāskarācārya studied mathematics. Bhāskarācārya and *Līlāvatī* loved each other. When the time of separation came at the end of the study, Līlāvatī urged Bhāskarācārya to marry her. But, since according to *Dharmaśāstra,* Līlāvatī was related to Bhāskarācārya by the relation of a sister (such a sister is called *guru-bhaginī, 'sister through the relation of one's guru* i. e. teacher), Bhāskarācārya declined the offer but at the same time promised her to write a book which would be titled on her name; and that book was *Līlāvatī.* The story is really romantic, fit to form the subject-matter of a romantic drama; yet it is without any evidence. Actually, the evidence of the family history given by Bhāskarācārya himself, viz. तज्जस्तच्चरणारविन्द युगल प्राप्त प्रसादः (cf. verses 61, 62 from गोलाध्याय quoted before) points to the contrary fact that he studied under his own father and that he had no other teacher than his own father. This legend prevails in the Māḷawā district.

It will be clear from the above discussion that the name Līlāvatī does not happen to be a proper name of a girl/lady related to Bhāskarācārya.

Then what is the explanation of the title *Līlāvatī* of the book ? The following two explanations may perhaps throw light on the

problem.

The word *Līlāvatī* is a fem. of the word *Līlāvat* which is a *taddhita* formation from the word *Līlā*. The word *līlā* signifies many meanings such as 'play, sport, amusement' and also 'charm, grace, beauty' etc. The latter meaings viz. 'charm, grace, beauty' etc. are more suitable in the present context since it is fem; and the derivative *Līlāvatī* will mean one 'possessed of charm, grace or beauty' etc. The word is also a fem. adjective; and it would then have a feminine substantive such as *pāṭī* referring to अंकगणित or पाटीगणित.

The word पाटी, it should be noted, actually occurs in the first verse of the text. The word *līlāvatī* together with the word *pāṭī* to be borrowed and supplied from verse no. 1 would form a phrase like लीलावती पाटी, meaning 'the arithmetic with a charm of its own', or if we accept the meaning of 'sport, play, pastime' etc. for the word लीला, the whole phrase would mean 'arithmetic, with ease, sport, play, pastime' etc. The whole title of the book would then be लीलावती पाटी.

Another explanation of the title लीलावती offered here is based on a certain way or custom or method of study in ancient India.

We know that the Brahmins in ancient India used to recite Vedas and all its six auxiliaries (i. e. षडङ्ग) daily as their sacred duty. But the task of reciting daily all this gigantic and bulky literature was an impossible one. They found out a way. What they did was to recite only the first line of the text concerned; it was a symbolic act of recitation, symbolising the remembrance and recitation of the whole text. It was decided that by reciting the first line of the text, the whole text was taken to have been recited fully. All the first lines of the sacred texts were then collected together, and the whole composition was entitled as ब्रह्मयज्ञ.

It will be seen from the text of ब्रह्मयज्ञ (Vide Appendix E) that each of its sentence reminds and refers to each of the Vedic, Brahmanical, Upaniṣadic texts, as also to the षडंगs of the Vedas.

Thus, the line अग्निमीळे पुरोहितम् refers to Ṛgveda; इषे त्वोर्जे त्वा to Vājananeyi Saṁhitā and so on.

But surprisingly enough, the science of mathematics is conspicuously absent from the list of sacred texts. The reasons may be that either mathematics at the times of the ब्रह्मयज्ञ composition was not developed as a pure science or was not accepted as one of the वेदाङ्ग's, or, that the science of mathematics did not have any authoritative text-book which could be referred to and recited daily. Be it as it may, but the fact is that the text of ब्रह्मयज्ञ does not contain a reference to any mathematical work in its times.

Taking clue, perhaps, from the practice and fact that a book can be referred to by its initially occuring passages, stanzas or words, Bhāskarācārya gave the title *Līlāvatī* to his work. The word *līlā* occurs in the stanza लीलागललुलल्लोल etc. It is from the initial word *līlā* in the stanza that Bhāskarācārya derived an adjectival form *līlāvat with the taddhita* suffix - *vat;* and since arithematic was called पाटीगणित or simply पाटी, the derivative *līlāvat* was transformed into its feminine as *līlāvatī.* And it is with this name लीलावती that Bhāskarācārya himself designated his work. We cannot say whether Bhāskarācārya intended his work to be included in and recited with the Vedic texts listed in the ब्रह्मयज्ञ text. Yet, that he titled his work by mentioning the first word of his text, after the fashion of ब्रह्मयज्ञ text, cannot be totally lost sight of.

An important point requires some explanation. We have said above that the initial word लीला in the verse has been used to name the whole work. But the problem is : in the text of लीलावती which has come down to us, the verse लीलागललुलल्लोल is not the first verse but occupies the 12th position, and the first place is occupied by the verse प्रीतिं भक्तजनस्य. How then can we take the word लीला and the verse in which it occurs as being in the beginning of the text ?

If we examine the arrangement of the text of लीलावती, we find the first section called 'the परिभाषा' deals with the different technical

terms, scales and the exchange-tables for the coins, weights, measures, length, area and volumes. This section therefore does not seem to be strictly mathematical since it does not state and involve any mathematical operations; the section seems to be included in the initial part as an *ā priori* acquaintance with the technical terms given in the later, strictly mathematical, part. The real text of लीलावती, as a text on pure mathematics begins actually from the verse लीलागल. We can, therefore, safely say that the word लीला is the first word of the लीलावती text.

Or, it is also possible that the tables relating to exchange of scales and measures in the परिभाषा section might have been part of an independent text and are included later on in the initial part of लीलावती for convenience of easy and ready reference. They could also as much have been appended at the end of the text, since they are not an indispensable part of strictly 8 mathematical operations of addition, subtraction etc.

7. *Literary qualities of the text :-*

The text of *Līlāvatī,* as we can easily see, is composed in verse and not in prose, as has been the practice of the ancient Indian Sanskrit scientists. The only motive behind the versitication seems to be that, since the knowledge was handed down orally, poetry or versitication proved to be of great value and convenience in oral traditions. In spite of the dry and purely theoritical nature of the science of mathematics, Bhāskarācārya seems to have tried to bring as much literary charm in his compositions as possible, and his versification does exhibit certain literary qualities and merits.

7.1 The first glaring quality of the composition that strikes one is the address of the teacher to the student, by the words बाले, सखे etc. In all there are in the present text five such words used as address to the disciple; they are : बाले, मतिमति, बालकुरंगलोलनयने, कल्याणिनि, सखे. All these adjectival addresses to the student show the love, affection and fondness tendered by the teacher for the students. Four of these adjectives are in the feminine and only one viz. सखे in the masculine. This does not mean that the students

included more girls or that Bhāskarācārya was really teaching a girl called लीलावती. It only means Bhāskarācārya took double advantage of the name लीलावती, which is the name of his work as well as which can also be used as a proper name of a lady. The name लीलावती occurs as the name of the book in verse no. 260 (लीलावतीह सरसोक्तिमुदाहरन्ती), as also as the name of a lady in verse no. 12 (अये बाले लीलावति).

From another point of view, the adjectives address the लीलावती as a book. The rules of mathematical operations given in the text are the real help to the students in solving the mathematical problems. The address to the book लीलावती would, therefore, mean something like : 'O लीलावती (= the text), solve the given problems contained in yourself.' Such an interpretation would then take Bhāskarācārya to a still higher level of poetic fancy from which he is viewing his own creation. It is like Cassius in SHAKESPEAR'S *Julijus Ceasar* speaking 'O Hands, speak for me' while assaulting Ceasar first.

The phrase बालकुरंगलोलनयने really exhibits poetic fancy and height.

7.2. Another characteristic style of Bhāskarācārya's composition is that while treating the students very fondly and affectionately, he at the same time challenges them to solve the mathematical problems. Thus, the phrases like यदि व्यक्ते... असि कुशला (Verse 12), यदि कल्या असि (Verse 16), जानासि चेत् (Verse 20), बुद्धेर्विवृद्धिर्यदि तेऽत्र जाता (Verse 22) and यदि घनेऽस्ति घना मतिः (26) throw a sort of encouraging challenge to the students; the challeges are neither worded in strong or rude words nor are meant to discourage them. They show Bhāskarācārya's affectionate attitude towards his disciples.

7.3 As a poet, Bhāskarācārya seems to be very much fond of the अनुप्रासालंकार. The following are some of examples of अक्षरानुप्रास from the thirty verses taken here for study.

7.3.1. Repetition of ल :- कोमलामलपदैर्लालित्यलीलावतीम् (Verse no. 1); लीलागललुलल्लोलकालव्यालविलासिने नीलकमलामलकान्तये (10A); बाले लीलावति (14); बाले बालकुरंगलोलनयने (16).

7.3.2. Repetition of द्ध :- विद्धि बुद्धेर्विवृद्धि (22);

7.3.3 Repetition of words :- मतिमति (12); कल्यासि कल्याणिनि (16); घनेऽस्ति घना (26).

These examples are collected only from the portion from verses 1 to 30. A thorough study of the whole text from literary point of view may throw more light on the literary qualities of Bhāskarācārya and may bring to light greater use of more अलंकार's. It may also be remarked that the ancient Indian scientists, including the Vedic sages were poets of a very high order; even while treating the dry, theoretical subjects like mathematics, they did not lose sight of the human and poetic element hidden in the very nature of man.

8. I am very happy that the Board of Studies in Sanskrit, University of Poona, Pune, has introduced Bhāskarācārya's *Līlāvatī* as a subject in the curriculm of the MA course in Sanskrit. The portion prescribed covers the eight mathematical operations stated in the first 30 verses of *Līlāvatī*. The subject of Sanskrit mathematics didnot uptill now find a place in the curriculum at any stage of higher education, and to introduce *Līlāvatī* was the necessity of time. Because, the real stage of development of a civilisation can only be known by the knowledge of its mathematical science. And if Indian civilisation is of a very high order, it could not attend that status without progress in mathematics. To know the stage of mathematical knowledge of a civilisation is just not information but is necessary to measure its greatness. I congratulate all the members of the Board of Studies in Sanskrit.

I am extremely thankful to all of them, expecially the chairman, Prof. N. N. KOLHAPURE, and Dr. V. N. JHA, the Director of the Centre of Advanced study in Sanskrit for entrusting the work of the translation of *Līlāvatī* to me. The text is to be taught from June 1992 itself. I had a very short time at my disposal to finish the work. But that I could finish the work within a period of one month is a matter of relief to me that I could justify the trust

the members of the BOS in Sanskrit put in me.

I must, however, mention one thing. Because of the short time at my disposal, some mistakes in the form of ommissions might have inadvertantly crept in in the work. The readers, I hope, will certainly pardon me for the mistakes.

I also thank the Veda Vidyā Mudraṇālaya for the prompt, efficient and decent printing.

Pune
15th May, 1992,
Nṛsiṁha Jayantī.

M. D. PANDIT

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॥ गणेशस्तुतिः ॥

Verse No. 1 :

प्रीतिं भक्तजनस्य यो जनयते विघ्नं विनिघ्नन् स्मृतः तं वृन्दारकवृन्दवन्दितपदं नत्वा मतंमाननम् । पार्टी सद्गणितस्य वच्मि चतुरप्रीतिप्रदां प्रस्फुटाम् संक्षिप्ताक्षरकोमलामलपदैर्लालित्य लीलावतीम् ॥ 1 ॥

Padapātha : प्रीतिम् । भक्तजनस्य । यः । जनयते । विघ्नम् । विनिघ्नन् । स्मृतः । तम् । वृन्दारकवृन्दवन्दितपदम् । नत्वा । मतंगाननम् । पाटीम् । सद्गणितस्य । वच्मि । चतुरप्रीतिप्रदाम् । प्रस्फुटाम् । संक्षिप्ताक्षरकोमलामलपदैः । लालित्यलीलावतीम् ॥ 1 ॥

Construction :-

यः विघ्नं विनिघ्नन् भक्तजनस्य प्रीतिं जनयते (इति) स्मृतः, तं वृन्दारकवृन्दवन्दितपदं मतंगाननं नत्वा (अहं) चतुरप्रीतिप्रदां प्रस्फुटां संक्षिप्ताक्षरकोमलामलपदैः लालित्यलीलावतीं सद्गणितस्य पाटीं वच्मि ॥

Translation :-

Bowing down to Lord Gaṇapati (lit. elephand - headed one), who is known (lit. remembered) to inspire reverence (lit. love, affection) (in the minds) of the devotees (because he) dispels (lit. kills, destroys) (all) difficulties, who is (lit. whose feet are) worshipped (lit. bowed) by the groups of gods, I lay down (lit. speak out) the rules (or methods) of pure mathematies of real numbers, (entitled) 'Līlāvatī full of interest' (lit charm) which creates interest (lit. love) in the skilled persons, (and) contains clear, (yet) brief, smooth-sounding, defect-less letters and words.

Notes :-

वृन्दारकवृन्दवन्दितपदः - वृन्दारकानां वृन्दैः वन्दितं पदं यस्य सः ।

मतंगाननः - मतंगस्य आननमिव आननं यस्य सः ।

चतुरप्रीतिप्रदा (१) प्रीतिं प्रददाति इति प्रीतिप्रदा (२) चतुराणाम् । चतुरेभ्यः प्रीतिप्रदा चतुर-प्रीति-प्रदा ॥

संक्षिप्ताक्षरकोमलामलपद -

(१) संक्षिप्तम् अक्षरम् संक्षिप्ताक्षरम् ।

(२) कोमलानि च अमलानि च कोमलामलानि ।

(३) संक्षिप्ताक्षराणि च कोमलामलानि संक्षिप्ताक्षरकोमलामलानि ।

(४) संक्षिप्ताक्षरकोमलामलानि पदानि यस्यां सा ।

(५) लालित्यलीलावती - लालित्येन युक्ता लालित्ययुक्ता । लालित्ययुक्ता लीलावती लालित्यलीलावती ।

पाटी - lit. method as in परिपाटी; here it means the arithmetic; पाटी = पाटीगणित.

स्मृतः = lit. remembered; but here 'known, famous.' The word can also be construed with the phrase विघ्नं विनिघ्नन् and the whole phrase would mean 'one, who destroys all obstacle by simply being remembered.'

सद् - गणित = सत् गणितम् सद्गणितम् । = pure mathematics of real numbers, as different from what in modern terminology called as the applied mathematics, and also mathematics of complex numbers.

Though the study of mathematics in India can be traced as far back as the Vedic times, it was used only as an auxiliary science, as a help to the sacrificial, astronomical and astrological studies. In that sense, mathematics has always been treated as an applied science and never seems to have developed as a pure science, until the time of Bhāskarārya. Bhāskarārya's *Līlāvatī* is the first known text-book on mathematics as a pure science in the history of ancient Indian mathematics. All other sciences like Astronony, Astrology, Sacrificial performances etc. have simply taken the help of mathematics, but no attempt seems to have been made to develop mathematics as an independent, pure branch of knowledge. It has always remained as a hand-maid of other sciences. It must, however, be mentioned that wherever mathematical methods and theories are used in other sciences, they exhibit a sufficiently higher knowledge of mathematical methods; cf. for example, the *śulba-sūtras* (cf. R. P. KULKARNI, *Cār Śulbasūtre* in Marathi, Mahārāshtra Rājya Sāhitya Sanskriti Maṇḍaḷ, Bombay, 1978) which use a good number of

higher geometrical methods; Or even the portion in Varāhamihira's *Bṛhatsaṁhitā,* dealing with the *cārādhyāyas* of the nine planets, or even *Āryabhaṭīya* of Aryabhata I, bear testimony to the fact that mathematics does not seem to have been neglected; on the contrary, mathematics seems to have been slowly progressing, enriched by various methods regarding the known and unknown quantities and areas and spheres. Yet, the fact remains that it has always remained an applied science and never developed as an independent, pure one. The importance and merit of Bhāskarācārya's works on arithmetic, algebra and geometry lie in the fact that he has tried to collect all the theories in these branches of mathematics together and give us a text - book on pure mathematics and not as a hand - maid of some other science. This is what is meant by the phrase सद् - गणित.

Why mathematics did not develop as an independent, pure science, we donot know. Perhaps it is possible that many text-books on the subject might have been in existence in the times of the Brāhmaṇas, Śulba-sūtras, Āryabhaṭa and/or Varāhamihira; but they or their traditions might have been lost. Secondly, it is also posible that the knowledge of pure mathematics might have been so much well-known and wide-spread in those, pre-Bhāskarīya times that they did not think it necessary to write a separate text-book on pure mathematics; the knowledge might have been utilised in day-today life by the people. Thirdly, it is also possible to imagine that the knowledge of pure mathematics might have been purposefully kept secret from common people for the purposes of sanctity and sacredness. Fourthly, it is also possible that because of the extremely abstract, and hence complicated and difficult nature of the subject of pure mathematics itself and all the theoretical methods therein, its study gradually was on the point of waning as the days went by and the subject was totally neglected and forgotten by the people, even by the scientists of the times. Whatever the reasons, there seems to be a very wide gap between the period when mathematics was used extensively on a wider scale in Astronomy, Astrology, Sacrificial geometry etc. and the one when suddenly, in the form

of Bhāskarācārya's *Līlāvatī* the subject, as a pure science, appears to have entered into the phase of mathematical renaissance in India.

It must, however, be stated that though the phrase सद्-गणित may refer to the branch of pure mathematics, no such phrase as असद्गणित is even seen to have been employed in the whole of mathematical literature to refer to the branch of applied mathematics. Actually, the ancient Indian mathematicians do not seem to have made any such distinction as 'pure' (i.e. सत्) and 'applied' (i.e. असत्) in mathematics.

As is well-known, mathematics includes the four main branches, viz. the arithmetic, algebra, geometry and trigonometry. Out of this पाटी or पाटीगणित, also called as व्यक्तगणित, refers to the arithmetic, which deals will different relations of numbers, both positive and negative, real and imaginary. The algebra, called बीजगणित or अव्यक्तगणित in Sanskrit, treats the relations of unknown quantities; it uses only unknown quantities like या, मा, का etc. The geometrical studies aim at studying the areas and volumes of the space. The geometrical studies, therefore, examine the entire space. The Sanskrit words for geometry are भूमिति or ज्यामिति (from which the latter word 'geometry' in English has been derived). The trigonometry forms part of geometry and is rendered as त्रिकोणमिती in Sanskrit.

The phrase सद्-गणित may also refer to the true and higher methods which bring out the relationships of numbers. It is just not sufficient to know the relations of numbers in terms of only addition and subtraction. Besides these simple relations, the numbers exhibit other relations also viz. the multiplication, division, squares and square-roots, cubes and cube-roots, fractions etc. Without knowing all these relations, it is not possible to understand the working of numbers in all their aspects and realms.

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Section 1 (The *Paribhāṣās*)

The ordinary language which is used in daily communication is insufficient to, or even incapable of meeting the needs of accuracy in scientific considerations. Every science, to be absolutely accurate and exact in enunciating its statements, principles and theories, requires a special type of language. Let us, for example, take the word *vṛddhi.* In daily communication, the word signifies the sense of 'growth, development' etc. from *vṛdh* 'to grow, develop' (*cf.* Pāṇinian *dhātupāṭha, vṛdha vardhane).* But when Pāṇini wants to use this word in his grammar, he finds it to be absolutely useless for conveying the grammatical sense of the *vṛddhi* of *i, u* into *ai, au.* Though, therefore, Pāṇini borrows the word from the ordinary speech of the people, he ascribes a special meaning to it in the *sūtra, vṛddhir ād āic,* 1.1.1. In other words, he *defines* the general word *vṛddhi* to give out a special, technical meaing. All the sciences have to follow similarly this type of exercise of ascribing new, technical meanings to the words of daily use in order that the theories of knowledge that they propound and explain must be very accurately and exactly worded and stated.

The rules of defining the simple words of daily use and investing them with definite, techincal and scientific meanings, which are different from the meanings the words convey in daily communication are called by the name *'paribhāṣā,'* as against the *bhāṣā* which refers to the general, traditional meanings accepted by the people in their daily communication.

The present section of *Līlāvatī* deals with the *paribhāṣas* of weights and measures. But it is to be remembered that Bhāskarācārya does not define in this sections all the techincal words in Mathematics. He has defined only those words which are used in weighing and measuring and the coins.

Verse - 2 वराटकानां दशकद्वयं यत् सा काकिणी ताश्च पणश्चतस्रः । ते षोडश द्रम्म इहावगम्यो

द्रम्मैस्तथा षोडशभिश्च निष्कः ॥ 2 ॥

Construction :- यत् वराटकानां दशकद्वयं सा काकिणी । ताश्च चतस्रः (काकिण्यः) पणः । ते षोडश (पणाः) द्रम्मः (इति) इह (पाटीगणितशास्त्रे) अवगम्यः । तथा च षोडशभिः द्रम्मैः निष्कः (भवति) ।

Translation :- What are Two Tens (i. e. twnenty) of *Varāṭakes,* (they make) one *Kākiṇī;* the four *Kākiṇīs* (make) one *Paṇa;* the Sixteen (Paṇas) (make) one *Dramma* (this is what is) to be understood here (in this science of *Pāṭīgaṇita);* and (lastly) by Sixteen *Drammas* one *Niṣka* (is to be understood).

The verse lists all the coins of the times. It gives the Table of equivalence of the unit coins used in the times of Bhāskarācārya. In terms of a Table, it is as follows :-

20 *varāṭaka* = *1 kākiṇī*
4 *kākiṇīs* = 1 *paṇa*
16 *paṇas* = 1 *dramma,* and
16 *drammas* = 1 *niṣka*

Verse 3 : तुल्या यवाभ्यां कथितात्र गुञ्जा वल्लस्त्रिगुञ्जो धरणं च तेऽष्टौ । गद्याणकस्तद्द्वयमिन्दतुल्यैः वल्लैस्तथैको धटकः प्रदिष्टः ॥ ३ ॥

Construction :-
इह गुञ्जा यवाभ्यां तुल्या कथिता । वल्लः त्रिगुञ्जः (कथितः) । ते च अष्टौ (वल्लाः) धरणं (कथितम्) तद्द्वयम् गद्याणकः (प्रदिष्टः) । तथा इन्द्रतुल्यैः वल्लैः एकः धटकः प्रदिष्टः ॥

Translation :- Here (i. e. in this Science) one *guñjā* is said to be equal to two *yavas;* one *valla* is (said to be) of three *guñjās;* one *dharaṇa* (is said to be of) the (above defined) eight *guñjās;* a pair of them (i. e. the *dharaṇa)* (is given to be equal to) one *gadyāṇaka;* a *dhaṭaka* is defined as equal to fourteen *vallas.*

The verse lists the weights (and not measures) used in weighing the valuable metals like gold and silver. The smallest weight is the *yava* which is the grain or granule of barley. In terms of a Table of equivalence, it is as follows :-

2 *yavas* = 1 *guñjā*
3 *guñjās* = 1 *valla*
8 *vallas* = 1 *dharaṇa*

2 *dharaṇas* = 1 *gadyāṇaka;* and
14 *gadyāṇakas* = 1 *dhaṭaka*

Note that the word *indra* signifies the number 14, as there are 14 *indras*. The word *indra* is called a word - number or a word-numeral; as against this, the word *catur-daśa* is called a number-word. It is a practice of the ancient Indian mathematicians to denote numbers by means of such words which signify a significand which has a permanently fixed number of things or objects. Thus, since 'the eyes' are always 'two', the Sanskrit word *netra* (and consequently all its synonyms) is used to signify the number '2' "From the beginning of the Christian era, the word-numerals with their various synonyms began to be used to avoid repetition of the same word and to keep rhythm of the ślokas[1]."

Verse 4 : दशार्धगुञ्जं प्रवदन्ति माषम् माषाह्वयैः षोडाभिश्च कर्षम् । कर्षैश्चतुर्भिश्च पलं तुलाज्ञाः कर्षं सुवर्णस्य सुवर्णसंज्ञम् ॥ ४ ॥

Pada-pāṭha दशार्धगुञ्जम् । प्रवदन्ति । माषम् । माषाह्वयैः । षोडशभिः । च । कर्षम् । कर्षैः । चतुर्भिः । च । पलम् । तुलाज्ञाः । कर्षम् । सुवर्णस्य । सुवर्णसंज्ञम् ॥

Construction :- दशार्धगुञ्जं माषं प्रवदन्ति । षोडशभिः च माषाह्वयैः कर्षम् (प्रवदन्ति) । तुलाज्ञाः चतुर्भिः च कर्षैः पलं (प्रवदन्ति) । सुवर्णस्य कर्षं सुवर्णसंज्ञम् (प्रवदन्ति)

Translation :- They define (Lit. speak of) *Māṣa* as half of ten *guñjās;* by sixteen *māṣas* (they define) *Karṣa;* the experts in balances (speak of) one *pala* by (the composition of) four *karṣas*; the *karṣa* (in the case) of gold has the name *suvarṇa* itself.

In terms of a Table, we get. 1/2 of 10 i.e. 5 *guñjās* = 1 *māṣa;* 16 *māṣās* = 1 *karṣa.* 4 karṣas = 1 *pala.* In the case of gold, 1 *karṣa* = 1 suvarṇa; that is to say, *karṣa* is equal to *suvarṇa.* The name *suvarṇa* is another term besides *karṣa.* Perhaps, these two different terms might have been used in different

1. *cf* A K BAG, *Mathematics in Ancient and Medieval India*, Chaukhamba Oriental Research Studies, No.16, Chaukhamba Orientalia, Varanasi, 1979, p. 60; for details, *cf Saṅketa-Koṣa* and Appendix D.

territories and have come to be used simultaneosuly in *Bhāskārācārya's* times and place.

Verse 5 : यवोदरैरंगुलमष्टसंख्यैर्हस्तोऽगुलैः षड्गुणितैश्चतुर्भि : । हस्तैश्चतुर्भिर्भवतीह दण्डः क्रोशः सहस्रद्वितयेन तेषाम् ॥ ५ ॥

Pada-pāṭha : यवोदरैः । अङ्गुलम् । अष्टसङ्ख्यैः । हस्तः । अङ्गुलैः । षड्गुणितैः । चतुर्भिः । हस्तैः । चतुर्भिः । भवति । इह । दण्डः । क्रोशः । सहस्रद्वितयेन । तेषाम् ॥ ५ ॥

Construction - अष्टसङ्ख्यैः यवोदरैः अङ्गुलम् (भवति) । षड्-गुणितैः चतुर्भिः अङ्गुलैः हस्तः (भवति) । इह चतुर्भिः हस्तैः दण्डः भवति । तेषां सहस्रद्वितयेन क्रोशः (भवति) ।

Translation :- One *aṅgula* is defined (lit. becomes equal to) by the *yavas* (lit. swollen, middle parts of *yavas*) numbering eight; the *aṅgulas* numbering four multiplied by six (give rise to) one *hasta*; four *hastas* make (lit become) one *daṇḍa* (and finally) two thousands of (the *daṇḍas*) (make) one *krośa*.

The verse enumerates the measurements of space i.e. length, breadth and height. The measurements in the form of a Table are :-

8 *yavas* = 1 *aṅgula*
24 *aṅgulas* = 1 *hasta*
4 *hastas* = 1 *daṇḍa* and
2000 *daṇḍas* = 1 *krośa*

Note the phrases षड्गुणितैः चतुर्भिः which is cuqual to 4 multiplied by 6 i.e. 24 and सहस्रद्वितय, which signifies a pair of thousands, i.e. two thousands.

Verse 6 : This verse continues the measurements of space; it especially defines the scales to be used for long distances.

स्याद् योजनं क्रोशचतुष्टयेन
तथा कराणां दशकेन वंशः ।
निवर्तनं विंशतिवंशसंख्यैः
क्षेत्रं चतुर्भिश्च भुजैर्निबद्धम् ॥ ६ ॥

pada-pāṭha स्यात् । योजनम् । क्रोशचतुष्टयेन । तथा । कराणाम् । दशकेन । वंशः । निवर्तनम् । विंशतिवंशसंख्यैः । क्षेत्रम् । चतुर्भिः । च । भुजैः । निबद्धम् ॥

Construction क्रोशचतुष्टयेन योजनं स्यात् । तथा कराणां दशकेन वंशः (स्यात्) । चतुर्भिः भुजैः निबद्धं च क्षेत्रं विंशतिवंशसंख्यैः निवर्तनं (स्यात्) ॥

Translation : Four *krośas* make (lit become) one *yojana;* in the same way, ten *hastas* (make) one *vaṁśa;* the area enclosed on (lit by) four sides, each being of 20 *vaṁśas* in length. (is to be called) a *nivartana.*

In the form of a Table,

4 *krośa* = 1 *yojana*

10 *hastas* = 1 *vaṁśa*

A four-sided figure, each side of 20 *vaṁśas* in length = 1 *nivartana.*

Verses nos. 5 and 6*ab* define only different measurements of length. The terms *aṅgula, hasta, daṇḍa, krośa, yojana and vaṁśa* therefore, relate only to one dimension, viz. length alone.

6*cd* however suddenly changes its level of dimensions and enters into two dimensional structures, viz. area enclosed on all sides by length and breadth. This *pāda* thus defines the measurement of a foursided figure in/on a plane. The unit of measurement given is *'nivartana';* the plane *(kṣetra)* or *area* enclosed within four sides of 20 *vaṁśas* in length each is called *'nivartana'*. The following figure will give an idea of a *nivartana* :-
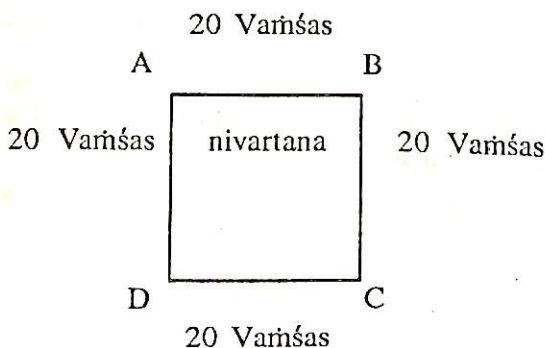
Fig. 1

The area enclosed by the four sides, AB, BC, CD and AD each of 20 *vaṁśas* in length, in the above figure no. 1 is called *nivartana;* a *nivartana* is not necessarily a square, it can be even a figure like the parallelogram. In Sanskrit philosophy, the space enclosed on four sides of the same or of different lengths is called *paṭākāśa* i.e. 'space within a *paṭa* or plane', referring to the two-dimensional space.

Verse No.7 : This and the following two verses nos. 8 and 9 define solid figures or measurements used in measuring the grains or some such things.

हस्तोन्मितैर्विस्तृतिदैर्ध्यपिण्डैः यद् द्वादशास्रं घनहस्तसंज्ञम् । धान्यादिके यद् घनहस्तमानं शास्त्रोदिता मागधखारिका सा ॥ ७ ॥

Pada-pāṭha : हस्तोन्मितैः । विस्तृतिदैर्ध्यपिण्डैः । यद् । द्वादशास्रम् । घनहस्तसंज्ञम् । धान्यादिके । यद् । घनहस्तमानम् । शास्त्रोदिता । मागधखारिका । सा ॥

Construction हस्तोन्मितैः विस्तृतिदैर्ध्यपिण्डैः यद् द्वादशास्र (भवति, तत्) घनहस्तसंज्ञम् (भवति) । धान्यादिके यद् घनहस्तमानं सा शास्त्रोदिता मागधखारिका (भवति) ।

Translation : The solid (figure) with length and breadth of the measure of one hasta (lit. measured by one hasta) which has twelve sides is known by the term *ghana-hasta*. The *ghana-hasta* measure which is (used) in the case of grains etc. is (the measure known as) *Khārikā* in the Magadha - country; it is mentioned (lit spoken) in the science/s.

vistṛti = *vistāra* = length (from *vi+√stṛ* to spread; *dairghya* = *dīrghatā* = breadth (from *dīrgha*); *piṇḍa* = a solid figure. *dvādaśāsra* = *dvādaśa + asra; asra* = side.

Thus we have that the *ghana-hasta* of Bhāskarācārya is the same as the *khārī* or *khārikā* measure used in the Magadha - country. The *ghana-hasta*, therefore, is a solid figure of one *hasta* in length, one *hasta* in breadth and one *hasta* in height. The following figure will illustrate the *ghana-hasta* measure :-
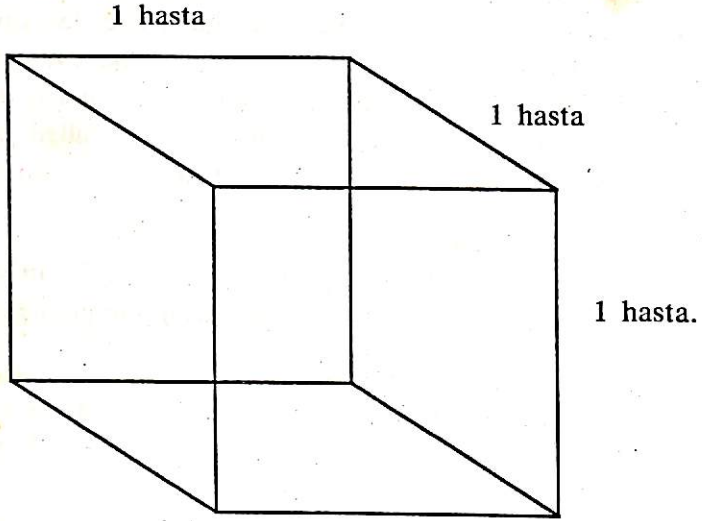
Fig. 2 : one *ghana-hasta* measure.

The following compounds should be noted :-
हस्तोन्मितैः - हस्तैः उन्मिताः हस्तोन्मिताः । तैः । उन्मित = उद् + मित from √ मा to measure. विस्तृतिदैर्ध्यपिण्डः = विस्तृति + दैर्ध्य + पिण्ड । विस्तृति = वि + स्तृति, from √ स्तृ + the fem kṛt suffix ति.
दैर्ध्य, from दीर्घ, दीर्घस्य भावः दैर्ध्यम् । विस्तृतिः च दैर्ध्यं च विस्तृतिदैर्ध्ये (if इतरेतर) or, विस्तृतिदैर्ध्यम् (if समाहार द्वन्द). विस्तृतिदैर्ध्याभ्याम् (or, विस्तृतिदैर्ध्येण) युक्तः पिण्डः विस्तृतिदैर्ध्यपिण्डः । तैः ॥ द्वादशास्रम् = द्वादश अस्राणि । अस्राः यस्यतत् ॥ घनहस्तसंज्ञम् = घनहस्ता संज्ञा यस्य तत् ॥

A *ghana-hasta* is euqual to one cubic *hasta* of length, breadth and height of one *hasta* each in measure. In Sanskrit, it is called *ghaṭākāśa* space enclosed within an earthen pot referring to three-dimensional space. The words घटाकाश and पटाकाश are used a number of times in Vedānta and Nyāya.

Verse 8 : द्रोणस्तु खार्याः खलु षोडशांशः
स्यादाढको द्रोणचतुर्थभागः ।
प्रस्थश्चतुर्थांश इहाढकस्य
प्रस्थांघ्रिराद्यैः कुडवः प्रदिष्टः ॥ ८ ॥

pada-pāṭha - द्रोणः । तु । खार्याः । खलु । षोडशांशः । स्यात् । आढकः । द्रोणचतुर्थ भागः । प्रस्थः । चतुर्थांशः । इह । आढकस्य । प्रस्थाङ्घ्रिः । आद्यैः । कुडवः । प्रदिष्टः ॥

Construction : द्रोणः तु खार्याः षोडशांशः (स्यात्) खलु । द्रोणचतुर्थभागः आढकः स्यात् । इह आढकस्य चतुर्थांशः प्रस्थः (प्रदिष्टः) । आद्यैः प्रस्थाङ्घ्रिः कुडवः प्रदिष्टः ॥

Translation :- (The measure) *droṇa* is, indeed, the 16th part of *khārī; āḍhaka* is the fourth part of *droṇa; prastha* is the fourth part of *ādhaka;* the *kuḍava* is defined by the ancients (lit. first) as the fourth part of *prastha.*

In the form of a Table, it is as follows:-
droṇa = 1/16 of *khārī*;
āḍhaka = 1/4 of *droṇa;*
prastha = 1/4 of *āḍhaka;* and
kuḍava = 1/4 of *prastha.*

In the inverse way, the measures are as follows:-
4 *kuḍavas* = 1 *prastha;*
4 *prasthas* = 1 *āḍhaka;*
4 *āḍhakas* = 1 droṇa; and
16 *droṇas* = 1 *khārī.*

The *kuḍava* is the smallest and the *khārī* is the highest measure. Though it is not stated explicitly in the verse, these measures, as we know from the tradition, were used in the case of liquids, like water, milk, oil etc. The *āḍhaka* for example was used in Varāhamihira's times to measure rains; cf Varāhamihira, Bṛhat Saṁhitā, 23.2 पञ्चाशत्पलमाढकमनेन मिनुयाज्जलं पतितम् । The measure of *āḍhaka* in Varāhamihira's times seems to be 50 *palas*. This *pala* however, is not the same *pala* as defined by *Bhāskarācārya* in verse no. 4 above. Datye's Almanac and Euphemeris of the year 1978 AD (Śka 1900; p.4) defines *ādhaka* as a volume of 60 sq. *yojanas* with a height of 100 *yojanas;* thus आढक = 60 x 100 =

600 cubic *yojanas. cf.* आढकप्रमाणम् । षष्टियोजनविस्तीर्णं शतयोजनमुन्नतम् । आढकस्य प्रमाणं तु देवमानेन गण्यते ।

The word *aṅghri* literally means the *pāda* or foot. But since the word *pāda* also signifies the one-fourth part of a metre, the word *aṅghri* is also taken in the sense of 1/4th part.

Verse No. 9 : पादोनगद्याणकतुल्यटंकैः
द्विसप्ततुल्यैः कथितोऽत्र शेरः ।
मणाभिधानं खयुगैश्च शेरैः
धान्यादिमानेषु तुरुष्कसंज्ञाः ॥ ९ ॥

pada-pāṭha : पादोनगद्याणकतुल्यटंकैः । द्विसप्ततुल्यैः । कथितः । अत्र । शेरः । मणाभिधानम् । खयुगैः । च । शेरैः । धान्यादिमानेषु । तुरुष्कसंज्ञाः ॥

Construction अत्र पादोनगद्याणकतुल्यटंकैः द्विसप्ततुल्यैः शेरः कथितः । खयुगैश्च शेरैः मणाभिधानम् (कथितम्) । (एताः) तुरुष्कसंज्ञाः धान्यादिमानेषु (उपयुज्यन्ते) ॥

Translation :- In this science, a *śera* is equal to seventy-two *taṅkas* (which are) equal to three-fourth of *gadyāṇaka.* With forty *śeras* (a measure of) one *maṇa* is counted (lit. is said). (These are) the Turkey names/terms (used in) measuring grains etc.

In terms of a Table, we have,
1 *taṅka* = 3/4 of *gadyāṇaka*
72 *taṅkas* = 1 *śera*
40 *śeras* = 1 *maṇa.*

A *gadyāṇaka* as we know from verse no. 3 is 8 *dharaṇas* or 24 *vallas* or 48 *guñjās.* The three-fourth of *gadyāṇaka* therefore will be equal to 6 *dharaṇas* or 18 *vallas* or 36 *guñjās* This is the measure of a *taṇka*. A *śera* therefore is equal to 72 X 6 *dharaṇas* or 72 X 18 *vallas* or 72 X 36 *guñjās.* The *maṇa* in turn will then be equivalent of 40 X 72 X 36 *guñjās*

Compounds:
पादोनगद्याणकतुल्यटङ्कैः = पादेन ऊनः पादोनः । पादोनः गद्याणकः पादोनगद्याणकः । पादोनगद्याणकेन तुल्यः पादोनगद्याणकतुल्यः । पादोनगद्याणकतुल्यः टंकः पादोनगद्याणकतुल्यटंकः । तैः ॥

द्विसप्त (= 2, 7) = 72 according to the axim अंकानां वामतो गतिः so also खयुग (युग ख) that is 40. युग, being always 4 in number comes to signify the number 4; and the word-numeral ख which means the sky, void signifies the number zero.

As Bhāskarācārya has stated here clearly, the terms *taṅka, śera* and *maṇa* are borrowed from the Turkish people, or in general from the Yavanas.
तुरुष्क = The Turkish people.

शेषा कालादिपरिभाषा लोकप्रसिद्धा ज्ञेया ।
padapāṭha : शेषा । कालादिपरिभाषा । लोकप्रसिद्धा । ज्ञेया ॥
Construction शेषा कालादिपरिभाषा लोकप्रसिद्धा (अस्ति) । (सा) ज्ञेया ॥
Translation : The remaining technical terms (for measuring) Time are well-known amongst the people and should be known from them.
The Yeraṇḍolkar-edition of *Līlāvatī* contains one more verse which is as follows:-

Verse 9 A : द्वंयकेन्दुसंख्यैर्धटकैश्च सेरः
तैः पञ्चभिः स्याद् धटिका च ताभिः ।
मणोऽष्टभिस्त्वालमगीरशाह -
कृतात्र संज्ञा निजराज्यपूर्षु ॥ 9A ॥

padapāṭha : द्वयंकेन्दुसंख्यैः । धटकैः । च । सेरः । तैः । पञ्चभिः । स्यात् । धटिका । च । ताभिः । मणः । अष्टाभिः । तु । आलमगीरशाहकृता । अत्र । संज्ञा । निजराज्यपूर्षु ॥ 9 A ॥

Construction : द्वयंकेन्दुसंज्ञैः धटकैः च सेरः (स्यात्) । तैः पञ्चभिः (शेरैः) च धटिका स्यात् । ताभिः अष्टभिः (धटिकाभिः) तु मणः (स्यात्) । अत्र निजराज्यपूर्षु आलमगीरशाहकृता संज्ञा (व्याख्याता) ।

Translation : By 192 धटकs, one शेर (is defined); by those very 5 शेरs one धटिका (is defined); with the 8 धटिकाs one मण (is calculated); these terms (lit this term) (are) promulgated (lit. done) by the emperor आलमगीर in the cities of his kingdom.

In the form of a Table, it is as follows :-

192 घटकs = 1 शेर
5 शेरs = 1 धटिका
8 धटिकाs = 1 मण.

द्वयंकेन्दु = द्वि + अंक + इन्दु; द्वि = २, अंक (lit. number) = 9 and इन्दु (lit. moon) = 1 And by the principle of अंकानां वामतो गति:[2] 291 is changed to 192.

The two verses viz. 9 and 9A seem to have been added later on; because history has no record of a Muslim ruler in the times of Bhāskarācārya. To which Muslim ruler the name आलमगीर or आलमगीरशहा refers is not known.

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2 For a detailed note on अंकानां वामतो गति: *cf M. D. PANDIT, Mathematics As known To The Vedas,* Vol. I (The Vedic Saṁhitās) Indian Books Centre, New Delhi, 1992.

Section 2

Verse No. 10 : लीलागललुलल्लोलकालव्यालविलासिने ।
गणेशाय नमो नीलकमलामलकान्तये ॥ 10 ॥

padapāṭha : लीलागललुलल्लोलकालव्यालविलासिने । गणेशाय । नमः । नीलकमलामलकान्तये ॥

Construction : लीलागललुलल्लोलकालव्यालविलासिने नीलकमलामलकान्तये गणेशाय नमः ॥

Translation :

Obeisance to lord Gaṇeśa who is (conspicuously) shining on account of the deadly (lit death-ly) cobra around his neek which is freely rolling and moving out of pleasure, and who bears a colour as stainless as the blue lotus.

Notes :-

a. लीला.... विलासिने

1. लुलति इति लुलन् ।
2. कालः इव व्यालः कालव्यालः ।
3. लोलः कालव्यालः लोलकालव्यालः ।
4. गले लुलन् गललुलन् ।
5. लीलया गललुलन् लीलागललुलन् ।
6. लीलागललुलन् लोलकालव्यालः लीलागललुलल्लोलकालव्यालः ।
7. लीला... व्यालेन विलसितुं शीलं यस्य सः
लीलागललुलल्लोलकालव्यालविलासी । तस्मै ।

b. नीलकमलामलकान्तये -

१. नीलम् कमलम् नीलकमलम् ।
२. नीलकमलम् इव अमला नीलकमलामला ।
३. नीलकमलामला कान्तिः यस्य सः । तस्मै ॥

The book contains two मंगलाचरण verses one in the beginning of the परिभाषाऽ (प्रीतिं भक्तजनस्य etc) and the other here at the beginning of the text containing the actual rules of mathematical operations.

Either one of the verses of मंगलाचरण is spurious, inserted afterwards, or the text of Līlāvatī, which has come down as one seems to have been originally in two parts; one part contained all the परिभाषाs regarding coins, weights, measures of length, area and volumes, and all the unstated, unexplained techniques required in mathematical operations (some of these are given in Appendix C at the end); the other part contained the rules for proper mathematical operations. We cannot say anything about this at this stage. The matter requires further research in the direction of bringing about a critical edition of *Līlāvatī*. But the fact remains that there are two मंगलाचरणश्लोकs in the present text of *Līlāvatī*.

Rule about the places of numbers

Verses 11 and 12 :

एकदशशतसहस्रायुतलक्षप्रयुतकोटयः क्रमशः ।
अर्बुदमब्जं खर्वनिखर्वमहापद्मशंकवस्तस्मात् ॥ ११ ॥
जलधिं चान्त्यं मध्यं परार्धमिति दशगुणोत्तरं संज्ञाः ।
संख्यायाः स्थानानां व्यवहारार्थं कृताः पूर्वैः ॥ १२ ॥

pada-pāṭha एकदशशतसहस्रायुतलक्षप्रयुतकोटयः । क्रमशः । अबुर्दम् । अब्जम् । खर्वनिखर्वमहापद्मशङ्कवः । तस्मात् । जलधिम् । च । अन्त्यम् । मध्यम् । परार्धम् । इति । दशगुणोत्तरम् । संज्ञाः । सङ्ख्यायाः । स्थानानाम् । व्यवहारार्थम् । कृताः । पूर्वैः ॥

Construction : तस्मात् । एकदशशतसहस्रायुतलक्षप्रयुतकोटयः अर्बुदम् अब्जम् खर्वनिखर्व महापद्मशङ्कवः जलधिम् च अन्त्यं मध्यम् परार्धम् इति सङ्ख्यानां स्थानानां दशगुणोत्तरं संज्ञाः पूर्वैः व्यवहारार्थम् कृताः ॥

Translation : For the purposes (of convenient representation of numbers) the predecessors (in the field of mathemataties) defined (lit made or coined) for mathematical operations the (following) terms of the places of numbers in that order : *eka, daśa, śata, sahasra, ayuta, lakṣa, prayuta, koṭi, arbuda, abja, kharva, nikharva, mahāpadma, śaṅku, jaladhi, antya, madhya and parārdha,* each succeeding (term) being ten times (of the preceding one)

Notes :-

Verses nos. 11 and 12 name and define the places of numbers.

It should therefore be borne in mind that what these terms signify is not the numbers but the places of numbers in writing. As is well-known, Indians and Europeans write from left to right both the words as well as the mathematical numbers. Yet, there is a difference in evaluating the places of the numerical as well as non-numerical constituents in the writing. While in writing the non-numerical entities, the order or position of the entities is taken as it is, in writing the numerical symbols, the values of the symbols are in descending order from left to right . That is to say, in the example राज्ञ: पुरुष: the word राज्ञ: is the first or prior and पुरुष: is the second or posterior. We understand their positions in the order in which they are spoken or written. In the case of numerical symbol like, say, 125, however, the value of the numbers written to the left is higher than that of those written to the right. Thus the value of 2 is greater than that of 5 and the value of 1 is grater than that of 2. In other words the value of 5 is prior to that of 2 and the value of 2 is prior to that of 1.

The word दशगुणोत्तरम् gives us the ratio by which the values of the numbers are greater or less; and the ratio is 10:1 from left to right, or 1:10 from right to left.

The present two verses list in all 18 places or what are called 'ranks' or 'levels'. Each succeeding rank or level is ten times higher than the preceding one. Thus, to represent in modern number-symbols with exponents,

1. *eka* = 1
2. *daśa* = 10 (i.e. *eka* X 10)
3. *śata* = 100 (i.e. *daśa* X 10)
4. *sahasra* = 1000 (i.e. *śata* X 10)
5. *ayuta* = 10,000 (i.e. *sahasra* X 10)
6. *lakṣa* = 100,000 (i.e. *ayuta* X 10)
7. *prayuta* = 1000,000 (i.e. lakṣa X 10)
8. *koṭi* = 10,000,000 (i.e.*prayuta* X 10)
9. *arbuda* = 100,000,000 (i.e. *koṭi* X 10)
10. *abja* = 1000,000,000 (i.e. arbuda X 10)

11. *kharva* = 10,000,000,000 (i.e. *abja* X 10)
12. *nikharva* = 100,000,000,000 (i.e. kharva X 10)
13. *mahāpadma* = 1000,000,000,000 (i.e. *nikharva* X 10)
14. *śaṅku* = 10,000,000,000,000 (i.e. *mahāpadma* X·10)
15. *jalaldhi* = 100,000,000,000,000 (i.e. *śanku* X 10)
16. *antya* = 1,000,000,000,000,000 (i.e. *jaladhi* X 10)
17. *madhya* = 10.000,000,000,000,000 (i.e. *antya* X 10)
18. *parārdha* = 100,000,000,000,000,000 (i.e. *madhya* X 10)

In terms of exponents or indics, we get the following representation. The exponent means 'the power' to which the number is raised. The symbol of the exponent is written on the upper side to the right of the number. Thus, in 10^2 the symbol 2 is 'the exponent'.

eka = 10^0
daśa = 10^1
śata = 10^2
sahasra = 10^3
ayuta = 10^4
lakṣa = 10^5
prayuta = 10^6
koṭi = 10^7
arbuda = 10^8
abja = 10^9
kharva = 10^{10}
nikharva = 10^{11}
mahāpadma = 10^{12}
śaṅku = 10^{13}
jaladhi = 10^{14}
antya = 10^{15}
madhya = 10^{16} and
parārdha = 10^{17}

The VS 17.2 notes the following terms : *eka, daśa, śata, sahasra, ayuta, niyuta, prayuta, arbuda, ny-arbuda, samudra, madhya, anta* and *parārdha* It thus mentions only 13 ranks; the names of the

ranks are also different from the present ones given by Bhāskarācārya[3].

Vyavahāra is taken here to mean 'the practice of mathematical operations' like addition, subtraction etc. The word can also be construed with संख्याया: स्थानानां व्यवहारार्थम् and the whole phrase would mean, 'for the use of the places of numbers'. It is however, better to take it adverbially to connected with *kṛtāḥ* meaning thereby, 'they are defined or named for practice or use in mathematics'

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3. for a detailed comparison, cf M D PANDIT,, *ibid* pp 25-30

Section 3

॥ अथ संकलितव्यवकलितयो: करणसूत्रं वृत्तार्धम् ॥

Translation : Now (follows) the rule for the operations of addition and subtraction; (the rule is) in half metre.

Verse 13 : कार्य: क्रमादुत्क्रमतोऽथवाऽङ्कयोगो यथास्थानकमन्तरं वा ॥
pada pātha : कार्य: । क्रमात् । उत्क्रमत: । अथ । वा । अङ्कयोग: । यथास्थानकम् । अन्तरम् । वा ॥
Construction : यथास्थानकम् क्रमात् अथवा उत्क्रमत: अङ्कयोग: कार्य: । (यथास्थानकम्) अन्तरं वा (कार्यम्) ॥

Translation : Addition of numbers should be done in direct or reverse order according to the ranks; (so also) the subtraction (of numbers, lit difference) (should be found out) (according to the ranks.)

Notes :-
अंकयोग: = अंकानां योग: ।
योग, from युज् to unite, yoke, add etc = addition. अन्तर (lit difference) = subtraction.
क्रम = regular or direct order. This and the next word उत्क्रम are relative words. If from left to right is taken as क्रम, from right to left will be उत्क्रम; if from right to left is taken as क्रम, then from left to right will be उत्क्रम. The two words are opposite to each other.

There is yet another consideration, of the order of the numerical and non-numerical entities. In the case of numbers, the number-word is written from left to right; but the place-value or rank of the number-symbols is in ascending order from the right. Thus in the number-word, पञ्चदश, the क्रमं of the words in the sense of writing from left to right will be पञ्च as first and दश as the second, while in the number-symbol for पञ्चदश, viz. 15, the क्रम of the symbols

1, and 5, in the sense of place-value i.e. from the lowest to the highest will be 5 and 1. The उत्क्रम order in the above cases will be oppósite to those explained above.

Since we are working here with numbers, we refer by the term क्रम to the order of the number-symbols; thus the क्रम in the case of the number-symbol 15 ·(for पञ्चदश) will refer to a journey from 5 towards 1; the उत्क्रम will refer to the journey from 1 to 5. Bhāskarācārya himself has called, as we shall see later, the number-symbol 5 as आदि and the number-symbol 1 as अन्त्य. Thus in any number-symbol (say, 16537) the right symbol (i.e.7) is आदि or आद्य and the left symbol (i.e.1) is अन्त्य or अन्तिम.

उत्क्रम will refer to the opposite order of क्रम, in whatever sense one may take the word क्रम. In number-symbol 16537, journey from 1 to 7 is उत्क्रम, according to Bhāskarācārya. We follow him.

स्थानक = स्थान = Rank, level, place, referring to the levels of *eka, daśa, śata* etc. enumerated in verses nos 11-12 above.

This verse gives the procedure to be followed in the case of the operations of summation and subtraction.

The following example will illustrate the working of the rule.

Verse 14 : अये बाले लीलावति मतिमति ब्रूहि सहितान्
द्विपञ्चाद्वात्रिंशत्त्रिनवतिशताष्टादश दश ।
शतोपेतानेतानयुतवियुताँश्चापि वद मे
यदि व्यक्ते युक्तिव्यवकलनमार्गेऽसि कुशला ॥

pada-pāṭha अये । बाले । लीलावति । मतिमति । ब्रूहि । सहितान् । द्विपञ्चद्वात्रिंशत् त्रिनवतिशताष्टादश । दश । शतोपेतान् । एतान् । अयुतवियुतान् । च । अपि । वद । मे । यदि । व्यक्ते । युक्तिव्यवकलनमार्गे । असि । कुशला ॥

Construction :-

अये बाले मतिमति लीलावति यदि (त्वं) युक्तिव्यवकलनमार्गे व्यक्ते कुशला असि, (तर्हि) एतान् द्विपञ्चद्वात्रिंशत्त्रिनवतिशताष्टादश दश (अंकान्) शतोपेतान् ब्रूहि । अपि च (एतान् अंकान्) अयुतवियुतान् मे वद ॥

Translation : O Young Intelligent Līlāvatī, if you are proficient in addition and subtraction in arithmetic, speak out (the total when) numbers, 2, 5, 32, 193, 18, 10 are added to 100; also speak

out when (they are) subtracted. (lit. separated) from *ayuta* i.e. 10,000

Notes :-

व्यक्त = व्यक्त गणित i.e. पाटीगणित i.e. arithmetic as opposed to अव्यक्त or बीजगणित. युक्ति, from युज् 'to unite, add' = addition.
व्यवकलन = subtraction.
मार्ग (lit way, path) = method, procedure.
त्रिनवतिशत = 3 + 90 + 100 = 193

1. *Procedure* : for addition we first put all the numbers according to their ranks. The numbers 2 and 5 belong to *ekam* rank i.e. are single digits, the numbers 32, 18 and 10 belong to *daśa* rank and the numbers 193 and 100 belong to the *śatam* rank. For addition, we have now, as the above rule in verse no. 13 states, two ways of doing the mathematical operation; one, by the *krama* - order and the other by the *utkrama* order. We first illustrate the *krama* method of addition. We put the numbers according to their ranks as follows :-

1.1 The *krama* way :-

śatam	*daśa*	*ekam*	
-	-	2	
-	-	5	
-	3	2	
1	9	3	
-	1	8	
-	1	0	
1	0	0	
-	2	0	step 1 (addition of numbers under *ekam* rank)
1	4	-	step 2 (addition of numbers under *daśa* rank)

	2			Step 3 (addition of numbers under *śatam* rank)
योग:	3	6	0	Step 4 (addition of all the results of all additions)

Explanation :-

Step 1 : We add the figures of the *ekam* rank; the total is 20, out of which 0 is the *ekam* rank and 2 is the *daśa* rank. We therefore, write 2 under *daśa* and 0 under *ekam* accordingly.

Step 2 : We add the *daśa* rank numbers; the total is 14; 4 belongs to *daśa* and and 1 to the *śatam* rank.

Step 3 : We then add all the *śatam* numbers, and the total is 2.

Step 4 : We then add all the results of the additions of all the three ranks; and the final total is 360.

It should be noted that if the total of single-digit numbers gives a number of two-digits, the higher rank should be moved to the left accordingly. Thus in step no.1 the total is 20 which is a two-digit number; hence we move 2 to the rank of *daśa* and keep the zero under *ekam* only. So also in step no.2, the total is 14, of which 1 belongs to the *śatam* rank and 4 to the *daśa* rank; we therefore, move 4 to the *śatam* rank. This is the meaning of the phrase यथास्थानकम्.

The process of addition can also be done by following the *utkrama* i.e. reverse way, that is to say, from left to right. It is as follows :-

1.1 the *utkrama* way :-

	śatam	*daśa*	*eka*	
	-	-	2	
+	-	-	5	
+	-	3	2	
+	1	9	3	
+	-	1	8	
+	-	1	0	
+	1	0	0	
	2	-	-	step 1 (addition of *s'atam* number)
	1	4	-	Step 2 (addition of *daśa* numbers)
		2	0	Step 3 (addition of *ekaṁ*)
योग:	3	6	0	Step 4 (addition of all ranks)

Explanation :

Step 1 : We start from the left rank of *śatam* and add all the *śatam* rank numbers. The sum is 2.

Step 2.: Then we add all the *dasá* rank numbers; the total is 14, out of which 4 belongs to *daśa* rank and 1 belongs to *śatam* rank we therefore, move or write 1 under the *śatam* rank.

Step 3 : We then add all the *ekaṁ* rank numbers the total is 20, out of which 2 belongs to *daśa* rank and 0 to the *ekam* rank;

We write accordingly and make the total. The final total is 360. We can do the totalling in both ways either from above to below or from below to above.

2. *Procedure for subtraction* :- We first put the numbers according to their ranks as in the following. We have to subtract the total 360 from the number *ayuta* i.e. 10,000. In the case of subtraction also, we can follow both the methods of *kram* and utkrama order of numbers. Of the number 360, 3 belongs to *śatam* place; 6 to *daśa* place and 0 to *ekam* place. Of the number 10,000, the number 1 belong to the *ayuta* place; and the rest of the zeroes, from left to right belong to *sahasra, śata, daśa and ekam* places respectively in that order.

2.1 The *utkrama* way :-

	ayuta	*sahasra*	*sātam*	*daśa*	*eka*	
	1	0	0	0	0	
	-	-	3	6	0	
	1	0	7	4	0	Step 1
		1	1	-		
	1	9	6	4	0	Step 2
वियोग :	0	9	6	4	0	Step 3

Explanation :-

Step 1: We have nothing to subtract from 1 of the *ayuta* rank and zero of the *sahasra* rank. We, therefore, keep them as they are. Since 3 of the *śatam* cannot be subtracted from the zero of the *śatam* we borrow one from the next higher rank of *sahasra* and subtract 3 (which is actually 300) and we get the remainder as 7. The one *sahasra* which we have borrowed from the *sahasra-rank* is therefore, written down below the *sahasra-rank*. We then subtract 6 from the zero of the *daśa* rank above. We follow the same

procedure of borrowing one unit from the next higher rank of *śatam* and subtract 6, the remainder is 4. We write the borrowed *śata* unit under *śatam*. The subtraction of the zeroes of the *ekam* rank of both the numbers gives out zero only.

Step 2 : There is no number for subtraction from 1 of the *ayuta* place. We, therefore keep it as it is.

We then subtract the borrowed unit 1 from the zero of the *sahasra* rank. Since it cannot be subtracted, we again borrow 1 *ayuta* unit, which is written below the number 1 of *ayuta* place. The subtraction then gives out the number 9.

The next subtraction of 1 from 7 gives us 6; and of zero from zero give zero only.

Step 3 : We finally subtract the borrowed unit 1 from 1 of the *ayuta* level; the rest of the figures remain as they are since there is nothing for subtraction. And the final result is 9640.

2.2 The *krama* way :-

	ayuta	*sahasra*	*śata*	*daśa*	*ekam*	
	1	0	0	0	0	
	0	0	3	6	0	
	1	0	7	4	0	step 1
	-	1	1	-	-	
	1 1	9 -	6 -	4 -	0 -	step 2
वियोग :	0	9	6	4	0	step 3

The same procedure of borrowing one unit from the next higher ranks, in case the subtracted number is greater, is followed and we get the same result. In *krama* way, we start from the right and go to left. The procedure, given above, is self-explanatory.

The units borrowed from the higher ranks in case the subtracted numbers are greater are also called 'units in hand' in modern termiology. There is no word for them in Sanskrit mathematics.

Though the rules for addition and subtraction are given, the actual procedures to be adopted in actual mathematical operations are nowhere stated in any of the books on Sanskrit mathematics. They are to be learnt from the teacher. They are thus handed down through *guru-śiṣya-paraṁparā* 'the teacher - steudent tradition' as it is called.[4]

Unlike in the case of the operation of addition, we cannot follow the procedure of subtraction in both ways, that is, either from above to below or from below to above. We must operate only from above to below and not from below to above.

It is also to be noted that the two process, of addition and subtraction are opposite to each other.

•••

4. Though, therefore, people assert or believe that they can learn without *guru* by simply reading the books on the subject of study, their belief does not hold good in the science of mathematics.

Section 4

गुणन प्रकार : – The operation of multiplication

गुणने करणसूत्रं सार्धवृत्तद्वयम् ।

Translation : Now, the rule for multiplication in two-and-half metres.

Verses 15, 16 and 17 (of which only half is given)

गुण्यान्त्यमङ्कं गुणकेन हन्यादुत्सारितेनैवमुपान्तिमादीन् ।
गुण्यस्त्वधोऽधो गुणखण्डयुक्तस्तैः खण्डकैः संगुणितो युतो वा ॥ 15 ॥
भक्तो गुणः शुध्यति येन तेन लब्ध्या च गुण्यो गुणितं फलं वा ।
द्विधा भवेद्रूपविभाग एवं स्थानैः पृथग्वा गुणितः समेतः ॥ 16 ॥
इष्टोनयुक्तेन गुणेन निघ्नोऽभीष्टघ्नगुण्यान्वितवर्जितो वा ॥ 17 ॥

pada-pāṭha गुण्यान्त्यम् । अङ्कम् । गुणकेन । हन्यात् । उत्सारितेन । एवम् । उपान्तिमादीन् । गुण्यः । तु । अधः । अधः । गुणखण्डयुक्तः । तैः । खण्डकैः । संगुणितः । युतः । वा ॥ 15 ॥
भक्तः । गुणः । शुध्यति । येन । तेन । लब्ध्या । च । गुण्यः । गुणितम् । फलम् । वा । द्विधा । भवेत् । रूपविभागः । एवं । स्थानैः । पृथक् । वा । गुणितः । समेतः ॥ 16 ॥
इष्टोनयुतेन । गुणेन । निघ्नः । अभीष्टघ्नगुण्यान्वितवर्जितः । वा ॥ 17 ॥

Construction : गुणकेन गुण्यान्त्यम् अङ्कम् हन्यात् । एवम् उत्सारितेन (गुणकेन) उपान्तिमादीन् (गुण्यान् हन्यात्) ॥ १ ॥
गुण्यः तु अधः अधः गुणखण्डतुल्यः (तं हन्यात् । तदनन्तरम्) तैः खण्डकैः संगुणितः युतो वा (गुणनफलं दास्यति) ॥ २ ॥
येन भक्तः गुणः शुध्यति तेन लब्ध्या च वा गुण्यः गुणितः फलं (दास्यति) ॥ ३ ॥ एवं रूपविभागः द्विधा भवेत् – स्थानैः पृथक् वा गुणितः समेतः (च गुणनफलं दास्यति) ॥ ४ ॥
इष्टोनयुतेन गुणेन अभीष्टघ्नगुण्यान्वितवर्जितः वा निघ्नः (स्यात्) ॥ ५ ॥

Translation :

One should multiply the multiplicand by the multiplier.

In this way, moving forward, one should multiply the next multiplicands from the penultimate one onwards. (1)

(By making suitable parts of the multiplicand and/or multiplier), the multiplicand (or the multiplier), equal to the sum of the parts, (should) be multiplied and (the result should) be added.(2)

The multiplicand, multiplied by the (two parts, viz.) divisor and the quotient of the multiplier, (gives out) the result (of multiplication); (3).

In this way, the division of the number (lit form of the number) is of two types : either on the basis of places or ranks (or on the basis of addition and/or multiplication) the multiplicand thus multiplied (by the two parts) (gives out the result) after adding (the two separate results) (4)

When the division of the number is suitably made, (the multiplicand) multiplied by the multiplier which (is obtained) by the subtraction or addition of a suitable number (gives out the result) when (after the multiplication) the two results are added or subtracted.(5)

Notes :

गुण्य (from √ गुण to multiply) = that which is to be multiplied, the multiplicand. गुण, गुणक that by which the multiplication is carried out. Thus in 3 multiplied by 4, the number 3 is गुण्य i.e. multiplicand and the number 4 is गुण or गुणक i.e. the multiplier.
उपान्तिम = उप + अन्तिम = next to the final; penultimate.

भक्त (from √ *bhaj* 'to divide') = when divided.
√ शुध् (lit to be pure) = to divide in such a way that the remainder is zero.
लब्धि (from √ लभ् 'to obtain') = the quotient.
इष्टोन = इष्ट+ऊन = less by a suitably desired number. In the division, for example of 12 by 4, the result 3 is the लब्धि; the remainder is zero; the division is, therefore, called शुद्ध (lit pure, complete)
फल = result of the operation.
रूप विभाग = division or splitting on the basis of the form.
The number, say 12, can be split up on two bases (i) *rūpa* i.e. form and (2) the *sthāna* i.e. rank or level.

The splitting on the basis of *rūpa* can be done in many or

rather infinite ways. For example

a) 12 = 1+11 = 2+10 = 3+9 = 4+8 = 5+7 = 6+6 etc.
b) 12 = 1X12 = 2X6 = 3X4 = 6X2 etc. or
c) 12 = 13-1 = 14-2 = 15-3 = 16-4 = 17-5 etc.
d) 12 = 24/2 = 36/3 = 48/4 = 60/5 = 72/6 etc.

Such splitting is called रूपविभाग.

स्थान विभाग = the splitting on the basis of स्थान can be done in only one way, i.e. as 1 and 2, in which 1 represents the rank of decimal place and 2 the digital place.

निघ्न, from नि + √ हन् irregular for निहत = multiplied. वर्जित = ऊन = less, subtracted. अन्वित, युक्त = added.

The three verses 15, 16 and 17 lay down in all six ways of the operation of multiplication. The methods will be illustrated from the example on multiplication given by Bhāskarācārya in the next verse no. 18.

Verse 18 : बाले बालकुरंगलोलनयने लीलावति प्रोच्यताम्
पञ्चत्र्येकमिता दिवाकरगुणा अंका: कति स्युर्यदि ।
रूपस्थानविभागखंडगुणने कल्यासि कल्याणिनि
च्छिन्नास्तेन गुणेन ते च गुणिता: जाता: कति स्युर्वद ॥ 18 ॥

padapātha बाले । बालकुरंगलोलनयने । लीलावति । प्रोच्यताम् । पञ्चत्र्येकमिता: । दिवाकरगुणा: । अंका: । कति । स्यु: । यदि । रूपस्थानविभागखण्डगुणने । कल्या । असि । कल्याणिनि । च्छिन्ना: । तेन । गुणेन । ते । च । गुणिता: । जाता: । कति । स्यु: । वद ॥

Construction यदि (त्वं) रूपस्थानविभागखण्डगुणने कल्या असि, (तर्हि) हे बाले बाल कुरंगलीलनयने कल्याणिनि लीलावति, प्रोच्यताम् - पञ्चत्र्येक मिता: दिवाकरगुणा: अंका: कति स्यु: ? (तथाच) तेन गुणेन च्छिन्ना: (तथाच तेन गुणेन) तेच (अंका:) गुणिता: कति स्यु: (इति) वद ॥

Translation :

O young, virtuous, Līlāvatī, having dangling eyes like those of a young deer, if you are expert in multiplication by splitting (the numbers) on the basis of रूपविभाग, and स्थानविभाग speak out : what will be the result (lit. numbers) if the number 135 is multiplied by 12, or tell (me) also, what number (lit. how many numbers) will emerge (lit. born) if the same (viz. those numbers which are

available after the multiplication) are divided and multiplied by the same multiplier (viz 12) ?

Note :-

The fourth *pāda* of the verse contains the word छिन्न which means 'divided' (from √ छिद् to divide). Since uptill now the procedure of division is not given, the example given in the *pāda* is not set for the present methods of multiplication. It seems to have been given for the next verse no. 19 which spells out the methods for division; see below. We explain only the first three *pādas*.

पञ्चत्र्येकमित = पञ्च + त्रि + एक + मित = 5-3-1 and by अंकानां वामतो गतिः = 135.

दिवाकर (= the sun) = 12

छिन्न (from √ छिद to divide) = divided.

We start solving the example. It is to be noted that the example can be solved by both the methods of क्रम and उत्क्रम order.

The verses nos. 15, 16 and 17 give us in all six methods of multiplication. Remember that 135 is गुण्य and 12 is गुणक.

1. first method (given in 15 ab):-

1.1. The क्रम - order (from right to left). :-

	Sahasra	śatam	daśa	ekam	
		1	3	5	(1 is अन्त्य; 3 उपान्तिम and 5 is आदि)
	x		1	2	
			6	0	(step 1 : 5 x 12)
		3	6		(step 2 = 3 x 12)
	1	2			(step 3 = 1 x 12)
योग :	1	6	2	0	= फल

Explanation :-

Step 1 : We first multiply 5 by 12. The result is 60, of which the number 6 is of *daśa* rank and 0 is of *ekam* rank. We, therefore, move 6 under *daśa*. This is the meaning of the word उत्सारित in the verse. The moving can be both ways, viz. from lower to higher rank or from higher to lower rank.

Step 2 : Multiply 3 by 12; it is 36; we move 3 under śatam rank.

Step 3 : Multiply 1 by 12; the result is 12; we move 1 to the higher rank of *sahasra*. Make the total and the result of multiplication is 1620 which is called फल

1.2 the उत्क्रम - order (from left to right.)

```
 135
 X 12
------
 12
  36   (the number is उत्सारित, moved to the left)
   60  (  ''        ''  )
------
1620 = फल
```

2. the second method (given in 15 cd) :-

In this case also both the क्रम order and the उत्क्रम order can be resorted to, though we have followed here the general traditional order of क्रम i.e. from right to left. One can also work out of his own the operation by the उत्क्रम order. The verse states the method of रूपविभाग of the गुण्य as well as गुणक. We consider one by one.

2.1 By resorting to रूपविभाग of the गुण्य :-

We split the गुण्य 135 into two parts as 70 and 65 so that 70 + 65 = 135. We now proceed as follows :-

70 X 12 = 840....i
65 X 12 = 780....ii

योग, = 1620 फल.

2.2 By resorting to the रूपविभाग of गुणक :-

We split up the गुणक 12 into 7 and 5, so that 7 + 5 = 12. Thus.

135 X 7 = 945....i
135 X 5 = 675....ii

योग: = 1620 फल

We can split the numbers in any other ways also and yet get the same results. The method can be represented by the formula $a(x+y)$, which is equal to $ax+ay$.

3. The third method (given in 16 ab) : This method relates to रूपविभाग. We make the रूपविभाग of both the गुण्य and गुणक on the basis of factorisation so that both of them give us the remainder

zero when divided by their respective factors. Thus, 135 = 45 X 3 (so that 135 ÷ 45, the remainder = 0; also 135 ÷ 3 the remainder is 0) and 12 = 6 X 2 (so that in 12 ÷ 6, or 12 ÷ 2 the remainder is zero) Now we proceed :- We may call this अवयवरूप विभाग.

3.1 अवयवरूपविभाग of गुण्य:-

(45 X 3) X 12
= (540 X 3)
= 1620 फल.

3.2 अवयवरूप विभाग of गुणक :-

135 X (6 X 2)
= 810 X 2
= 1620, फल.

4. The fourth method (given in 16 cd):-

This method relates to the स्थानविभाग of the गुण्य and गुणक, that is splitting the numbers on the basis of the ranks. We can, therefore, split up 135 as 1, 35 or 13,5 and 12 as 1,2.

4.1 स्थानविभाग of गुण्य:-

Sahasra śatam daśa ekam.

		1,	3	5	
		x	12		
		4	2	0	(step 1 : 35 x 12)
		1	2		(step 2 = 1 x 12)
योग :	1	6	2	0	= फल

4.2 स्थानविभाग of गुणक :-

Sahasra śatam daśa ekam.

		1,	3	5	
		x	1,	2	
		2	7	0	(step 1 : 35 x 12)
	1	3	5		(step 2 = 1 x 12)
योग :	1	6	2	0	= फल

5. The fifth method (given in 17 ab):-

The half-verse 17ab gives two methods - one which we may call as the इष्टोन method to be followed by addition (अन्वित) of the result and the other, which we may call as the इष्टयुक्त method to be followed by subtraction (वर्जित) of the results.

5.1 The इष्टोन - method (इष्ट + ऊन)

What is to be done is that from either of the two catergories of गुण्य and गुणक, a suitable number is to be subtracted; the two numbers are then multiplied. Since the number is subtracted, the results of the multiplication are to be added.

5.1.1 The इष्टोन from the गुण्य:-

To illustrate we subtract the number say, 75 from 135. We then multiply the new number (135-75) 60 as well as the number 75 which is subtracted and add the results. Thus,

135-75 = 60.

```
    75
  X 12
------
    60
   84
------
   900......i

and 60
  X 12
------
    00
   72
------
   720......ii
```

We then add 900 to 720 and get 900 + 720 = 1620 फल.

5.1.2. The इष्टोन from the गुणक:-

We subtract 3 from 12 and multiply 135 separately by 3 and (12-3) 9, then add the results thus,

12-3 = 9
135
X 3

15
9
3

405....i

135
X 9

45
27
9

1215...ii

we then add results (i) and (ii) and get,

405
1215

1620 फल.

6. The sixth method (given in 17ab).

We call this method as इष्टयुक्त method of multiplication. The results arrived at in the final are to be subtracted from each other. We can add any desired number to both the गुण्य and गुणक and then mulitiply.

6.1.1. इष्ट युक्त of गुण्य :-

We add suitably the number 40 to 135 and multiply the इष्टयुक्त number as well as the new number by 12; we get (135+40=) 175 as the new number. Thus,

40
X 12

00
48

= 480......(i)

Then we multiply the new number 175 and the इष्टयुक्त number by 12; and we get,

```
      175
     X 12
  ---------
       60
      84
     12
  ---------
  =  2100.... (ii)
```

We then subtract 480 from 2100 and get,

```
      2100
     - 480
  ---------
  =   1620, as the फल.
```

6.2.2. इष्ट युक्त of the गुणक:-

We add suitably the number 13 to 12 and make the multiplier as 25. We then proceed as follows :-

```
      135
     X 25
  ---------
      125
     75
    25
  ---------
    3375.....(i)
```

We then multiply the गुण्य 135 by the number 13 which is suitably added to 12 and proceed as

```
      135
     X 13
  ---------
       65
      39
     13
  ---------
    1755....(ii)
```

We then subtract (ii) from (i) and get,

```
       3375
      -1755
  ----------
       1620, as the फल.
```

One can see that whichever method, out of the six methods one follows for the operation of multiplication, one gets - and must get - the same result.

●●●

Section 5

॥ भागहारे करणसूत्रम् ॥

(Now) the rule for the operation of division (of numbers).

Verse 18 : भाज्याद्धरः शुध्यति यद्गुणः स्यात्
अन्त्यात् फलं तत् खलु भागहारे ।
समेन केनाप्यवर्त्य हार -
भाज्यौ भवेद् वा सति सम्भवे तु ॥ 18 ॥

papadātha :
भाज्यात् । हरः । शुध्यति । यद्गुणः । स्यात् । अन्त्यात् । फलम् । तत् । खलु । भागहारे । समेन । केन । अपि । अपवर्त्य । हारभाज्यौ । भवेत् । वा । सति । सम्भवे । तु ॥

Construction :-
यद्गुणः हरः अन्त्यात् भाज्यात् शुध्यति तत् खलु भागहारे फलम् स्यात् । सम्भवे तु सति केनापि वा समेन हारभाज्यौ अपवर्त्य (फलं) भवेत् ॥

Translation:-

(The multiplier) because of which the divisor fully divides the final dividend (i.e. the number which is to be divided) is indeed the answer in the operation of division. Or, (secondly) if it is possible, the answer can also be found (lit. can be there) by abbreviating the dividend and the divisor by some common (divisor or denominator.)

Notes :-
भाज्य =.the number to be divided, the dividend
हर, हार = the diviser, the denominator; शुध to divide fully, so that the remainder is either zero or some positive number.
गुण = गुणक = multiplier.
भागहार = भागाकार = operation of division.
अपवर्त्य, from अप + $\sqrt{}$ वृत् to abbreviate by dividing by a comman factor; सम = common factor.

Explanation :-

First Method - To illustrate the working of the process of division, let us take the example given by Bhāskarācārya in verse no 17 ab before. The भाज्य is 1620, the भाजक is either 135 or 12; we now proceed. First we divide the भाज्य viz 1620 by 135 :-

```
135 ) 1620 ( 12
     - 135
     ------
        270
        270
     ------
        000
```

The multiplier 1 in the quotient (called लब्धि or फल) 12, when multiplied with 135 which is the भाजक divides fully the अन्त्य number 162. The remainder of the subtraction of 135 from 162 is 27. The two numbers of the remainder 27 then form the *śatam* and *daśa* rank for the last digit zero of the भाज्य viz 1620 and forms the number 270; now this is the भाज्य for the next step. When divided by 135, the multiplier or quotient turns out to be 2 which forms the digital place for 1. The whole quotient viz. 12 therefore, is the answer or फल. of the division.

So also, when we take 12 as the divisor, the quotient or answer is 135.

Second Method :-

Let us illustrate the operation of division by the second method spelled out in verse no. 18 cd in which, first, we have to find out a common factor which will divide fully both the भाज्य and the भाजक. We write both the categories in the following way with 1620, the भाज्य, in the numberator and 12, the भाजक, in the denominator:-

$$\frac{1620}{12}$$

The common factor which will divide both is obviously first 2, and we have the abbreviated form as

$$\frac{810}{6}.$$

The next common factor for both of them is again 2; we divide both of them by 2 and we have

$$\frac{405}{3}$$

At this stage, we can again have the common factor which will divide both the denominator and the numerator; the common factor is 3 and we have as the final answer,

$$\frac{405}{3} = 135, \text{ the फल.}$$

This method is very useful when we face illustrations of division of big number with four or more ranks.

Rule for finding out whether a number is divisible by 2,3 or 5 :-

(a) rule for divisibility by 2 : The rule to find out whether any number is divisible by 2 or not is : if the given number has the figures 2, 4, 6, 8 and 0 at the digit rank on the right, the number is certainly divisible by 2. Thus, the numbers say 152, 7234, 10336, 120568, 228990 are definitely divisible by 2, while the numbers with figures 1,3,5,7,9 in the digit place are not divisible by 2. Thus, 153, 7235, 10337, 120569, 228991 are not divisible by 2.

(b) rule for divisibility by 3 : The rule to find out the possibility of the division of a number by 3 is : if the sum of all the numbers in the given number is divisible by 3, then the whole number is divisible by 3. For example, the sum of all the numbers in the numbers 78, 150, 3150, 93153 etc. is divisible by 3; hence all these numbers are definitely divisible by 3. Contrarily, the sum of all the numbers in 53, 179, 5179 etc. is not divisible by 3; hence they are not divisible by 3.

(c) rule for divisibility by 5 : The rule to find out whether a given number is divisible by 5 or not is : if the number 5 and /or zero occupy the digit place in the number, the number is divisible by 5. If this is not the case the number is not divisible by 5. Thus, the numbers 45,70,140,1375,10795 etc. have zero and 5 in their

digital places; they are, therefor, divisible by 5. The numbers on the other hand like 66, 72, 4307 etc. do not satisfy the above condition. They are, therefore, not divisible by 5.

It is to be noted, however, that these rules about finding out whether the numbers 2, 3 and 5 are factors of a given number or not are nowhere explicitly stated in any Sanskrit texts on mathematics. The rules are traditionally orally handed down by the teacher to the student.

●●●

Section 6

॥ वर्गे करणसूत्रं वृत्तद्वयम् ॥

Now the rule for the operation of finding out squares in two verses (lit. metres). वर्ग = square of a number. Thus 2 X 2 = 4. 4 is called the वर्ग of 2.

Verse 19 :- समद्विघात: कृतिरुच्यतेऽथ
स्थाप्योऽन्त्यवर्गो द्विगुणान्त्यनिघ्ना: ।
स्वस्वोपरिष्टाच्च तथाऽपरेऽङ्का:
त्यक्त्वान्त्युमत्सार्य पुनश्च राशिम् ॥ 19 ॥

padapātha : समद्विघात: । कृति: । उच्यते । अथ । स्थाप्य: । अन्त्यवर्ग: । द्विगुणान्त्यनिघ्ना: । स्वस्वोपरिष्टात् । च । तथा । अपरे । अङ्का: । त्यक्त्वा । अन्त्यम् । उत्सार्य । पुन: । च । राशिम् ॥

Construction:-
समद्विघात: कृति: उच्यते । अथ अन्त्यवर्ग: स्थाप्य: । स्वस्वोपरिष्टात् च तथा अपरे अङ्का: द्विगुणान्त्यनिघ्ना: (स्थाप्या:) । अन्त्यम् त्यक्त्वा पुन: च उत्सार्य राशिम् (कुर्यात्) ।

Translation :- Product of two equal numbers is called the कृति (i.e. square). First the वर्ग (or square) of the final number is to be written (or found out; lit to be established) Then numbers equal to the product of the next number with twice the final numbers (should be written) at the top; then leaving the final number, and moving again forward to the next, make the total.

Note :-

सम = the same, equal.
कृति = वर्ग = square.
अन्त्य = final from the right, (which becomes first from the left.)
द्विगुणान्त्यनिघ्न = multiplied with twice the final number.
स्वस्व + उपरिष्टात् = at the head of the respective numbers.
उत्सार्य = moving the ranks.
राशिम् = total, sum, addition.

(1) *first method* :
Verse 19a gives the first method for finding the square; it is very simple. It says, just multiply a number with its own self and we get the square. Thus, to work out the example of the number 9 given in verse 21 below, we find that the square of 9 is = 9 X 9 = 81. This simple method is applicable in the case of small, simple, single-digit numbers from 1 to 9. Besides the method of direct multiplication Bhāskarācārya also gives other methods to find out the squares of higher numbers consisting of 2,3,4 etc. digits. It is given in verse no. 19bcd.

(2) *second method*:-
Step 1 : First find out the square of the final number.
Step 2 : Multiply the final number by 2 and then with the next, pen-ultimate number.
Step 3 : Find out the square of the next, pen-ultimate number.
Step 4 : Take the total of all these results, keeping in tact their ranks.

Proceed in this way to the next numbers until we come to the end and the final result.

2.1 Illustration : Let us take the example of the number 14 given in verse no. 21 below.
Step 1 : Take the square of the final of the number 14, which is 1. Thus the square of 1 is 1 X 1 = 1 only.
Step 2 : Multiply the final 1 by 2 and then by 4, which is the pen-ultimate number. The product is 1 X 2 X 4 = 8
Step 3: Take the square of the next, pen-ultimate 4 which is 4 X 4 = 16
Step 4 : Add all these numbers keeping them in their proper ranks. Thus,

$$
\begin{array}{rl}
14^2 = & 1 \\
& +\ 08 \\
& +\ 016 \\
\hline
\end{array}
$$

196, which is the फल.

To take another example of still higher, three-digit number viz.

297 from the verse no. 21 itself. We first find out the square of 29 and then of 297. We proceed step by step.

29^2 = 4 अन्त्यवर्ग
+ 36
+ 081 उपान्त्यवर्ग
= 841

At this stage now, the number 29 becomes the अन्त्य with reference to 7, and we then proceed.

	841	अन्त्यवर्ग
+ -	406	29 X 2 X 7
+ -	49	उपान्त्यवर्ग ·
योग	88209	

We proceed in this way in the case of any higher numbers, splitting them up into their different ranks.

This method splits up 297 first into 29 and 7 and then 29 into 2 and 9. Thus, 2 is अन्त्य with reference to 9 and 29 is अन्त्य with reference to 7.

This method of Bhāskarācārya is based on the स्थानविभाग of the given number. What we have to do is to split the given number on the basis of its ranks and then apply this method to get square.

The verse no 19 thus gives us in all two methods of finding out the square of a given number. The first method consists of straight-a-way multiplying the given number with itself. This is what is called समद्विघात in the verse. The second method consists of splitting the given number on the basis of the ranks of its constituents.

Verse No. 20 :-

खण्डद्वयस्याभिहतिर्द्विनिघ्नी
तत्खण्डवर्गैक्ययुता कृतिर्वा ।
इष्टोनयुग्राशिवधः कृतिः स्यात्
इष्टस्य वर्गेण समन्वितो वा ॥ २० ॥

padapāṭha खण्डद्वयस्य । अभिहतिः । द्विनिघ्नी । तत्खण्डवर्गैक्ययुता । कृतिः । वा । इष्टोनयुग्राशिवधः । कृतिः । स्यात् । इष्टस्य । वर्गेण । समन्वितः । वा ॥

Construction :-

वा, खण्डद्वयस्य द्विनिघ्नी अभिहतिः तत्खण्डवर्गैक्ययुता (सती) कृतिः (स्यात्) । इष्टोनयुग् राशिवधः इष्टस्य वर्गेण समन्वितो वा (कृतिः स्यात्) ॥

खण्डद्वय - two parts; splitting into two parts.

द्विनिघ्नी -multiplied by 2.

अभिहतिः - हतिः = multiplication, product.

युक्, युत = added.

वध = हति = multiplication, product.

इष्ट = any desired, suitable number.

Translation :-

I. Twice the product of the two parts (of the given number) added to the squares of the two parts gives out (lit. becomes) the square of a number (20 ab)

Or again,

II. The square of a number can be also had (lit. is) by totalling the square of the desired, suitable number with the product of the two numbers obtained by adding and subtracting a suitable number from the given number.

Notes :-

This verse no. 20 states two more methods of finding out the square of a given number. Read with the foremer verse no. 19 the total number of methods of squaring laid down by Bhāskarācārya is four. The method given in 20ab does not basically differ from the one given in verse no. 19 above.

Explanation :-

20 ab : First split up the given number into two suitable parts. Then multiply those parts mutually as well as by 2. Add to this result the squares of the two parts. The final total is the result.

To borrow the examples from verse no. 21 The given number is 9; split this number into 2 parts, say 3 and 6, so that 3 + 6 = 9 Then multiply these parts by 2, which gives out 3 X 6 X 2 = 36. Then make the squares of the two parts and add them to the above result. We have then $3^2 = 9$ and $6^2 = 36$. Take

the total of all these results and we have 36 + 9 + 36 = 81 which is the square of 9.

20 cd : Add and subtract any desired number with and from the given number. Then multiply the new numbers mutually. Add to this result the square of the desired number which is originally added and subtracted. The result is the square of the given number.

To work out this process on the given number 9. Add say 5 to 9 The total is 14. subtract the same 5, which is the desired number, from 9 which is (9-5) = 4; we have now the two numbers viz. 14 and 4. Multiply them; the result is 14 X 4 = 56 Add to this the square of the desired number viz 5. which we have added and subtracted; the square is 25. Add this 25 with 56 and we have (56 + 25 =) 81 which is the square of the original, given number 9.

The third method given in 20ab consists of splitting the given number on the basis of its रूपविभाग (and not स्थानविभाग) as is done in verse no. 19 above) The रूपविभाग of a number gives us an expression in which two numbers are added. or subtracted or multiplied or divided; by addition. thus 9 = 4 + 5 or 14 = 10 + 4 etc. We then follow the method and get the result.

This method of रूपविभाग is based on an algebraic method of finding out the square of two unknown quantities added together. The algebraic method is enunciated by Bhāskarācārya in his own book on algebra entitled बीजगणित. This method follows the pattern of the algebraic method, viz. $(a+b)^2 = a^2+2ab+b^2$.

The foruth method of finding out the square is based on the simple, algebraic equation, viz. $a^2 = a^2-b^2+b^2$, in which a is the given number for squaring and b is the suitably added and subtracted number and we have,

$$a^2 = (a+b) \times (a-b) + b^2.$$

Verse no.21 :

सखे नवानां च चतुर्दशानाम् ।
ब्रूहि त्रिहीनस्य शतत्रयस्य ।
पञ्चोत्तरस्याप्ययुतस्य वर्गम्
जानासि चेद् वर्गविचारमार्गम् ॥ २० ॥

papadpāṭha : सखे । नवनाम् । च । चतुर्दशानाम् । ब्रूहि । त्रिहीनस्य । शतत्रयस्य । पञ्चोत्तरस्य । अपि । अयुतस्य । वर्गम् । जानासि । चेत् । वर्गविचारमार्गम् ॥

Construction : सखे, वर्गविचारमार्गं जानासि चेत् (तर्हि) नवानां चतुर्दशानां त्रिहीनस्य शतत्रयस्य च (अपि च) पञ्चोत्तरस्य अयुतस्य अपि वर्गं ब्रूहि ॥

Translation :-

O dear one, if you know well the way to find out (lit. think of) the squares, then speak out the squares of 9, 14, 297 and 10005.

Notes :-

त्रिहीन शतत्रय - three hundred less by three, i.e. 297.
The three examples of 9, 14 and 297 are already explained before. One can work out for himself the square of 10005.

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Section 7

॥ वर्गमूले करणसूत्रं वृत्तम् ॥

The rule for the operation of finding out the square root.

Verse no. 22 :

त्यक्त्वान्त्याद् विषमात् कृतिं द्विगुणयेन्मूलं समे तद्धृते
त्यक्त्वा लब्धकृतिं तदाद्यविषमाल्लब्धं द्विनिघ्नं न्यसेत् ।
पङ्क्त्यां पंक्तिहृते समेऽन्त्यविषमात्त्यक्त्वाप्तवर्गं फलम्
पङ्क्त्यां तद्द्विगुणं न्यसेदिति मुहुः पङ्क्तेर्दलं स्यात् पदम् ॥ २२ ॥

padapāṭha : त्यक्त्वा । अन्त्यात् । विषमात् । कृतिम् । द्विगुणयेत् । मूलम् । तद्हृते । त्यक्त्वा । लब्धकृतिम् । तदाद्यविषमात् । लब्धम् । द्विनिघ्नम् । न्यसेत् । पङ्क्त्याम् । पंक्तिहृते । समे । अन्त्यविषमात् । त्यक्त्वा । आप्तवर्गम् । फलम् । पङ्क्त्याम् । तद्द्विगुणम् । न्यसेत् । इति । मुहुः । पङ्क्तेः । दलम् । स्यात् । पदम् ॥

Construction :-

अन्त्यात् विषमात् (अंकात्) कृतिं त्यक्त्वा । मूलं द्विगुणयेत् । तद्हृते समे । लब्धकृतिं त्यक्त्वा तदाद्यविषमात् । लब्धं द्विनिघ्नं पङ्क्त्यां न्यसेत् । पङ्क्तिहृते समे । अन्यविषमात् आप्तवर्गं फलं त्यक्त्वा पंक्त्यां तद्द्विगुणं न्यसेत् । इति मुहुः (कुर्यात्) । (तदा) पंक्तेः दलं पदं स्यात् ॥

विषम - odd.
सम - even.
अन्त्य - last number at the left.
मूल - square-root.
द्विगुण - to double.
लब्ध/लब्धि - the quotient.
त्यक्त्वा (from $\sqrt{}$ त्यज्) - after subtracting.
आप्त-वर्ग - the square of the quotient.
दल (lit. leaf) - half.
पद - root, meaning both the square-root and the cube-root.

Translation :-

Subtracting the square from the last odd number, the (number which is the) square-root should be doubled. (Step no. 1) (This doubled

number should be written separately in a column called पंक्ति).

Divide the even number by this (double root)... (Step no. 2)
Subtract the square of the quotient (thus obtained from step no. 2 above) from the next odd number (Step no.3)
The quotient (thus obtained from step no 3 above) should be doubled and written in the column of the पंक्ति. (step no. 4)
(The two doubled roots written in the पंक्ति - column should then be added according to their ranks. This new number will then be the divisor of the next number in the example)... (step no. 5)
Divide the next even number by (the numbers in) the पंक्ति... (step no. 6)
Subtract the square of the quotient thus obtained from the next odd number (step no. 7)
This process should be repeated (until the remainder of the subtractions is zero)... (step no. 8)
The half of the पंक्ति will then be the square-root of the original, given expression.

Explanation:-

Let us take the example given in verse no. 23 below, which gives in all six examples, viz. 4, 9, 196, 81, 88209 and 100010025. We choose neither the simplest nor the biggest number, but the medium ones, viz. 196 and 88209.

Before proceeding to solve the examples proper, an important point requires to be remembered.

The whole given expression should be divided into and marked by signs of odd (विषम) and even (सम). The vertical line will indicate the विषम number and the horizontal bar will indicate सम number. We start marking the numbers from the right-hand number. Thus, in 196 the right-hand 6 is विषम, 9 is सम and 1 is again विषम as |-|
196

In 88209 the numbers 9,2 and the last 8 are विषम while 0 and pen-ultimate 8 are सम as |-|-|
88209. We now proceed to find out the square roots of 196 and 88209; first we take up 196. Thus,

भाजक	। - । 196	मूलम्	पंक्ति
1	1	1	2
(1x2=) 2	09		
	8	04	08
(4x2=) 8	16	14	28
	16	मूल root	28 ÷ 2 = 14, the root

Step 1 : Take the final विषम viz. 1 and find out the square root. The root is 1 itself. We write it in the column of मूल; subtract the product of 1 X 1 from the विषम 1; the remainder is zero. Double the root 1; it is now 2 which will be the divisor for the next number, which is सम, in the expression. We put this 2 under the column of भाजक and in the column of पंक्ति.

Step 2 - We now divide 9 by 2. The quotient is 4 which we will right in the column of मूल. The remainder of the subtraction 9-8 is 1.

Step 3 - Take down the next विषम 6 and the भाज्य is 16. Double the मूल viz. 4 which will be (4X2)8 and write it in the column of पंक्ति as well as भाजक.

Step 4. : Subtract from 16 the square of the मूल 4 and we have 16 - (4 X 4) = 0. The मूल, therefore is 14.

This method of Bhākskarācārya gives us a technique also for checking whether our result is correct. The technique is : Add according to ranks the numbers in the पंक्ति column and divide the total by 2. If the division gives out a quotient equal to the square-root in the column of मूल the result is correct. Thus, 28 ÷ 2 = 14 which is the same as the square-root in the column of मूल.

There is a certain step which is implied in this method. First the marking of विषम-सम is to be resorted to. The steps are not stated or defined in the text; they are to be learnt from the teacher itself. And when all the विषम-सम numbers are exhausted, and the remainder is zero, the process stops. Secondly, the विषम - place is the वर्ग-स्थान at which stage in the process, the वर्ग of the number in the पंक्ति is to be subtracted; simple division is not to be resorted to. The सम- place is अवर्ग-स्थान.

Example No. 2 : 88 20 9 (marked | - | - |)

भाजक	भाज्य	मूलम्	पंक्ति
	| - | - |		
2	88 20 9	2	2x2 = 4
	4		
4	48	9	9x2 = 18
	36		
	122	7	58
	-81		7x2 = 14
58	410	मूलम् =	594
	406	297	मूलम् = 594 ÷ 2
	49		
	49		= 297 = पंक्तिदल
	00		

Only the उत्क्रम order (i.e. from left to right) is to be followed in the case of finding out the वर्गमूल. The क्रम order (right-to left) does not work.

Verse no. 23 :

मूलं चतुर्ण्णां च तथ नवानां
पूर्वं कृतानां च सखे कृतीनाम् ।
पृथक् पृथग् वर्गपदानि विद्धि
बुध्देर्विवृध्दिर्यदि तेऽत्र जाता ॥

padapāṭha : मूलम् । चतुर्ण्णाम् । च । तथा । नवानाम् । पूर्वम् । कृतानाम् । च । सखे । कृतीनाम् । पृथक् । पृथक् । वर्गपदानि । विद्धि । बुद्धेः । विवृद्धिः । यदि । ते । अत्र । जाता ॥

Construction :-

हे सखे, यदि ते बुद्धेः विवृद्धिः अत्र जाता, (तर्हि) चतुर्ण्णाम् तथा नवानाम् (तथा) च पूर्वम् कृतानाम् कृतीनाम् च पृथक् पृथक् वर्गपदानि विद्धि ॥

Translation:-

O dear one, if your intelligence has quite developed in this science (lit. here) then know separately the square-roots of 9, 14 and of all the squared numbers calculated (lit. done) before (in the previous verses.)

Notes :-

The two examples of the previous squares viz. 196 and 88209 are already explained.

●●●

Section 8

॥ घने करणसूत्रं वृत्तत्रयम् ॥

(Now) the three verses (lit metres) stating the rules for finding out the cube of a number.

Verse no. 24 :-

समत्रिघातश्च घनः प्रदिष्टः
स्थाप्यो घनोऽन्त्यस्य ततोऽन्त्यवर्गः ।
आदित्रिनिघ्नस्तत आदिवर्गः
त्र्यन्त्याहतोऽथादिघनश्च सर्वे ॥ 24 ॥
स्थानान्तरत्वेन युता घनः स्यात् ॥ 25a ॥

Padapāṭha :-
समत्रिघातः । च । घनः । प्रदिष्टः । स्थाप्यः । घनः अन्त्यस्य । तत : । अन्त्यवर्गः । आदित्रिनिघ्नः । ततः । आदिवर्गः । त्र्यन्त्याहतः । अथ । आदिघनः । च । सर्वे । स्थानान्तरत्वेन । युताः । घनः । स्यात् ॥

Construction :-
समत्रिघातश्च घनः प्रदिष्टः । अन्त्यस्य घनः स्थाप्यः । ततः अन्त्यवर्गः आदित्रिनिघ्नः (स्थाप्यः) । ततः आदिवर्गः त्र्यन्त्याहतः (स्थाप्यः) । अथ आदिघनः (स्थाप्यः) । सर्वे च स्थानान्तरत्वेन युताः घनः स्यात् ॥

Translation:- Multiplication of three equal numbers is called घन (i.e.cube); (first) find out (lit. place) the घन of the अन्त्य number, then the square of the अन्त्य multiplied by three times the आदि; is to be written; then the square of the आदि multiplied by three times the अन्त्य is to be noted; and lastly the घन of the आदि is to be written. Then all totalled together according to their proper ranks represent the घन.

Notes :-
समत्रिघात - multiplication of three equal sums; समे त्रयः समत्रि । समत्रयाणां घातः समत्रिघातः । अन्त्य - the last from right and the first from left. आदि - the last from left and the first from right.

Explanation :-

This verse gives in all two methods of finding out the cube of a number. First method :- It consists of a simple multiplication-process of a sum three times; thus the घन of 9 = 9 X 9 X 9 - 729. This is given in verse 24a.

Second Method :- This method is explained in 24 bcd. It is more suitable for numbers having two or more digits or ranks.

Step 1 : Find out the cube of the अन्त्य.

Step 2 : Find out the square of the अन्त्य; multiply it by 3 and then by the next आदि number.

Step 3 : find out the square of the आदि, then multiply it by 3 and then by the अन्त्य number.

Step 4 : Find out the cube of the आदि.

Proceed in this way until all the ranks in the given numbers are exhausted. Then,

Step 5 : Add all these results; this will be the घनफल of the given expression.

Let us work out the two methods in the case of the examples given in verse No. 27. We take the number 27 which is equal to त्रिघन = घन of त्रि.

First method :- 27 X 27 X 27 =

```
  27            729
x 27           x 27
-----         ------
 4             1458
 28            5103
  49          ------
              19683   which is the फल.
-----
 729   (by the method of squaring)
```

Second method :- In 27, 2 is the अन्त्य and 7 is the आदि.

Step 1 : घन of अन्त्य	= 8	= 8
Step 2 : Product of square of अन्त्य, आदि and 3	= $3X2^2X7$	= 84
Step 3 : product of square of आदि, अन्त्य and 3	= $3X2X7^2$	= 294
Step 4 : घन of आदि	= 7X7X7	= 343
Step 5 : Total	= फल	= 19683

This method splits up the given number on the basis of *sthāna* or the rank.

The verse 25a requires to be borrowed for completing the meaning of verse no. 24.

Verse No. 25 :

स्थानान्तरत्वेन युता घन: स्यात्
प्रकल्प्य तत्खण्डयुगं ततोऽन्त्यम् ।
एवं मुहुर्वर्गघनप्रसिद्धा -
वाद्याङ्कतो वा विधिरेष कार्य: ॥ 25 ॥

padapāṭha :

स्थानान्तरत्वेन । युता: । घन: । स्यात् । प्रकल्प्य । तत्खण्डयुगम् । तत: । अन्त्यम् । एवम् । मुहु: । वर्गघनप्रसिद्धौ । आद्याङ्कत: । वा । विधि: । एष: । कार्य: ॥

Construction :-

तत्खण्डयुगं प्रकल्प्य तत: अन्त्यं (च प्रकल्प्य) । एवं वर्गघनप्रसिद्धौ आद्याङ्कत: तत: अन्त्यम् (प्रति इति क्रमेण) वा एष: विधि: कार्य: ॥

Translation :-

Splitting the (given number on the basis of रूप and स्थान) into two and fixing the अन्त्य, (all the results thus available by the method of cubing given in verse no.24) added together represent (lit. are) the cube of the number. In (finding out) the square as well as the cube the procedure can also be optionally started from the आद्य number (i.e. the first from the right hand).

Notes :-

स्थानान्तरत्वेन युता: :- Added by arranging the results in proper order of places or ranks.

तत्खण्डयुग - Two parts of the given number on the basis of स्थानविभाग and रूपविभाग.

This verse no. 25 should be read together with verse no. 24 above. The word युता: in 25 will have सर्वे from 24 as the कर्ता. The present verse lays down again two more methods for finding out the घन. The third method is stated in 25b and the fourth one is given in 25d.

Third method : The third method is to be applied in the case of numbers with more than two ranks or स्थानs. Let us take the example of finding out the घन of the घन of 5. i.e. finding out the घन of 125.

We divide the number into two parts, 12 and 5. Just as 1 is the अन्त्य and 2 is the आद्य in 12, the number 12 is the अन्त्य and 5 is the आद्य in the example 125. We can also split the number 125 into two parts as 1 and 25 with 1 as the अन्त्य and 25 as the आदि. This is what is meant by the phrase प्रकल्प्य तत्खण्डयुगं ततोऽन्त्यम् in the verse. We now proceed.

(A) घन 125 with parts 12 and 5 :

First we have to find out the घन of 12, which is as follows :-

A.1. $12^3 = 1^3+3.1^2.2+3.1.2^2+2^3 = 1+6+12+8.$

Arranged according to their स्थानs, 8 will be the right hand number, then 12, then 6 and lastly 1 which will be left-hand number.Thus,

$$\begin{array}{r} 1 \\ +\ 6 \\ +\ 12 \\ +\ \ \ 8 \\ \hline 1728 \end{array}$$

1728 is the घन of 12.

A.2. We then proceed to find out the घन of 125. 12 is अन्त्य and 5 is आदि. Thus

$125^3 = 12^3 + 3.12^2.5 + 3.12.5^2 + 5^3$

We arrange then in the proper order of place-value and we get,

$$\begin{array}{r} 1728 \\ +\ 2160 \\ +\ \ \ 900 \\ +\ \ \ \ 125 \\ \hline 1953125 \end{array}$$

योग = 1953125 = घन.

The fourth method stated in 25d lays down that the operation of cubing can also be carried out by beginning from the आद्य or आदि number which is at the right. All the former methods start from the अन्त्य at the left. As in squaring, we can, therefore, adopt any order - either of क्रम or उत्क्रम. In the उत्क्रम order however, the

places of ranks of the results, while adding, will have to be shifted to the right. In क्रम order we shift the ranks to the laft.

(B) घन of 27 by क्रम - order :-
We start from the आदि 7.

$27^3 = 7^3 + 3.7^2.2 + 3.7.2^2 + 2^8$

$= 343 + 294 + 84 + 8$

Now, we arrange them in their proper ranks and add :- (Shift to the left by one rank) :-

```
        343
      + 294
      + 84
      + 8
योग =  19683 = फल.
```

We can also work out the उत्क्रम order in the case of higher numbers like 125 etc.

Verse No. 26 :-
खण्डाभ्यां च हतो राशिस्त्रिघ्नः खण्डघनैक्ययुक् ।
वर्गमूलघनः स्वघ्नो वर्गराशेर्घनो भवेत् ॥ 26 ॥

padapātha : खण्डाभ्याम् । च । हतः । राशिः । त्रिघ्नः । खण्डघनैक्ययुक् । वर्गमूलघनः । स्वघ्नः । वर्गराशेः । घनः । भवेत् ॥

Construction :-
खण्डाभ्यां च हतः त्रिघ्नः राशिः खण्डघनैक्ययुक् (घनः भवेत्) । वर्गमूलघनः स्वघ्नः वर्गराशेः घनः भवेत् ॥

Translation :- The two parts (into which a number is split up on the baiss of रूपविभाग) multiplied with each other with the number and also by 3 and added to the total of the each of the cubes of the two parts results in the घन of the number; the घन of the square root of a number multiplied by itself becomes the घन of the square of the number.

Notes :-
त्रिघ्न - multiplied by 3.
खण्डघनैक्ययुक् - added to the total of the घन of the two parts.
स्वघ्न - multiplied by itself, it gives the square. राशि = number.

Explanation :- The verse no. 26 states two more methods of finding out the घन of a number. Together with the four methods, stated before in the previous verses nos. 24 and 25, the total number of methods of finding out the घन i.e. cube of a number amounts to 6. These are thus 6 methods of cubing given by Bhāskarācārya.

Fifth Method :-

If we want to find out the घन of a number which is split into two parts on the basis of the रूपविभाग we should multiply the parts with the number, then multiply it by 3 and add the number thus obtained to the two घनs of the two numbers. To illustrate, let us take the number 27 given as an example to work out in verse no. 27 below. We divide it into two parts viz. 15 and 12, so that 15 + 12 = 27. This is रूपविभाग and not स्थानविभाग. We then multiply 15 X 12 with 3 as well as the original number 27. The result is 14580. We then find out the घन of the two parts viz. 15 and 12, as respectively 3375 and 1728. We add the three results, viz.

14580
+ 3375
+ 1728
= 19683 which is the घन of 27.

It must be remembered that in totalling the three results viz. 14580, 3375 and 1728, we must not shift or move the places or ranks of the numbers, because the splitting of 27 is made on the bases of रूपविभाग and not on स्थानविभाग. The स्थानविभाग of 27 will be 2 (representing दशस्थान) and 7 (representing एकं स्थान.)

Sixth Method :- The sixth method deals with the process of cubing on a different level. The level is of squares. In all the examples and methods stated above, the starting level is of the first exponent or index or प्रथमघात. In the present method the initial stage for finding out the cube is of the second exponent or index or द्वितीय घात or वर्ग. This method gives us a technique by which we can immediately transform a square into its own cube. To illustrate let us take the original, given number as 9 which is itself a square

to find out its घन.

The method is : find out the square root of the given number, make the घन of the square root; and find out the square of the घन of this square root. The result is the घन of the original, given square. It thus deals with direct change from one transformational level to another.

Let us take the example of 9 given in verse no.27 below. It is a square. Thus, $9 = 3^2$.

To find out the घन of 9, reduce the square to its root, which is 3. Make the घन of 3 which is 27, and finally find out the square of the घन 27, which is 729. This is the घन of 9 or 3^2.

The first method of cubing is based on the simple process of multiplication of a number with itself three times; Thus, घन of $a = a \times a \times a$.

The second method is based on the algebraic formula of cubing the expression (a+b). thus, $(a+b)^3 = a^3 + 3\, a^2b + 3ab^2 + b^3$.

The third method also follows the above algebraic formula which is the basis for the second method above. But the numbers involved in this method have more than two ranks. Thus 125 has three ranks. This method requires the cubing in parts into which the big number is divided on the basis of स्थानविभाग.

The fourth method states that both the orders of क्रम and उत्क्रम can be followed in cubing a number. This rule, as we have noted before, is applicable in the case of finding out the squares also. In finding out the वर्ग and घन therefore, the order of numbers - from either left to right or right to left - does not matter at all. One can start from any end.

The fifth method is based on the रूपविभाग of the number, which can also be done in two ways; either to represent the given number as the sum-total of two numbers, or to represent it as the difference of two numbers. Thus just as the रूपविभाग of 27 is 15 + 12 (as we have done before) it can also be as 30-3 = 40-13 etc. etc. The only difference is that while in the case of the second method, there is the स्थानविभाग, in the present case it is रूपविभाग.

The method can be represented by the algebraic formula :
$a^3 = (x+y)^3$........a रूपविभक्त into x+y
$= x^3 + y^3 + 3xy\ (x+y)$,
which formula is the same as the one stated in the second method above.

The sixth method involves the direct transformation of a square into a cube.

The algebraic formula involved in this method is :

$$(a^2)^3 = (a^3)^2.$$

All these formulas are stated, explained and proved by Bhāskarācārya in his work on बीजगणित.

Verse no. 27 :- नवघनं त्रिघनस्य घनं तथा
कथय पंचघनस्य घनं च मे ।
घनपदं च ततोऽपि घनात् सखे
यदि घनेऽस्ति घना भवतो मतिः ॥ 27 ॥

padapātha :
नवघनम् । त्रिघनस्य । घनम् । तथा । कथय । पंचघनस्य । घनम् । च । मे । घनपदम् । च । ततः । अपि । घनात् । सखे । यदि । घने । अस्ति । घना । भवतः । मतिः ॥

Construction :- हे सखे, यदि घने (विषये) भवतः घना मतिः अस्ति, (तर्हि), नवघनम्, त्रिघनस्य घनम्, तथा च पञ्चघनस्य घनम् मे कथय । (अपि च) ततः अपि घनात् घनपदम् (कथय) ॥

Translation :- O friend, if your intelligence is uninterrupted in (the subject of finding out) the cubes, tell me the cube of 9, of the cube of 3, the cube of the cube of 5 and all the cubes from all these cubes.

Some of the examples from this verse are already. explained before. One can work out the rest for himself.

●●●

Section 9

॥ अथ घनमूले करणसूत्रम् वृत्तद्वयम् ॥

Now, the rule in two stanzas for finding out the cube-root.
The two verses should be read together.

Verse no. 28 : आद्यं घनस्थानमथाघने द्वे
पुनस्तथान्त्याद् घनतो विशोध्य ।
घनं पृथक्स्थं पदमस्य कृत्या
त्रिघ्न्या तदाद्यं विभजेत् फलं तु ॥ 28 ॥

Verse no. 29 : पङ्क्त्या न्यसेत् तत्कृतिमन्त्यनिघ्नीं
त्रिघ्नीं त्यजेत्तत् प्रथमात् फलस्य
घनं तदाद्याद् घनमूलमेवं
पङ्क्तिर्भवेदेवमतः पुनश्च ॥ 29 ॥

padapāṭha : आद्यम् । घनस्थानम् । अथ । अघने । द्वे । पुनः । तथा । अन्त्यात् । घनतः । विशोध्य । घनम् । पृथक्स्थम् । पदम् । अस्य । कृत्या । त्रिघ्न्या । तदाद्यम् । विभजेत् । फलम् । तु ॥ २८ ॥

पङ्क्त्याम् । न्यसेत् । तत्कृतिम् । अन्त्यनिघ्नीम् । त्रिघ्नीम् । त्यजेत् । तत् । प्रथमात् । फलस्य । घनम् । तदाद्यात् । घनमूलम् । एवम् । पङ्क्तिः । भवेत् । एवम् । अतः । पुनः । च ॥ २९ ॥

Construction : - आद्यम् घनस्थानम् (भवति) अथ द्वे अघने (स्थाने भवतः) । पुनः तथा अन्त्यात् घनतः घनं विशोध्य । पृथक्स्थं पदं (न्यसेत्) । अस्य (पदस्य) त्रिघ्न्या कृत्या तु तदाद्यं विभजेत् ॥ २८ ॥

फलं तु पङ्क्त्यां न्यसेत् । तत्कृतिम् अन्त्यनिघ्नीं त्रिघ्नीं तत्प्रथमात् त्यजेत् । फलस्य घनं तदाद्यात् त्यजेत् ।

एवं घनमूलं पङ्क्तिः भवेत् । एवम् अतः पुनश्च (कुर्यात्) ॥ २९ ॥

Translation :- The आद्य is (to be marked as) घनस्थान; the next two are अघन (स्थान); find (lit. finding) out the घनमूल from the अन्त्य; the result should be written (lit. put) separately. By multiplying the square of the result by 3, one should divide the next number (28);

The result of (this division) is to be put in the पङ्क्ति; multiply the square root of the result by 3, subtract it from the next (lit. first) number; then subtract the cube root from the next आद्य number. In this way., the पंक्ति will represent the घनमूल; (continue) similary again onwards.

विशोध्य (वि + शुध्), dividing and subtracting.
कृत्या - Inst. sing. fem. of कृति, 'square.'

These are the following steps in the operation :-
Step. 1 : Divide the given number into घन - parts and अघन - parts. The marking should be from the right i.e. आद्य.
Step. 2 : Take the घनमूल of the अन्त्यघन.
Step. 3 : Subtract is from the अन्त्यघन.
Step. 4 : Take the next आद्यांक against the result.
Step. 5 : Multiply the square of the first घनमूल by 3.
Step. 6 : Divide with this the next आद्यांक. The division should see that the remainder is more than the घन of the numbers 2 and 3. Otherwise, further subtraction and division will not be possible.
Step 7 : Multiply the square of the present quotient with the previous घनमूल and with 3.
Step. 8 : Divide the next आद्यांक by this result of multiplication. Take the remainder.
Step 9 : Subtract from this the घन of the आद्य घनमूल.

Continue the process until the remainder of the subtraction is zero.

Example : Bhāskarācārya has not given any example. We can, however, take any of the cubes found out previously in verses nos, 24, 25 and 26. Let us take first the simple example of 19683, to work out the method of finding out the cube root.

We put the number and mark the घनस्थानs and अघनस्थानs. We start from आद्य i.e. from the right. Thus, in 19683 the numbers 3 and 9 are the घनस्थानs. All others are अघनस्थानs..... step. 1.

We then find the cube root from the अन्त्य घनस्थान, viz. 19; ... step. 2

पंक्ति	19683	घनमूलम्
2	- 8	27
$2^2 x3 = 12$	116	
	- 84	
$7^2 x2x3 = 294$	328	
	- 294	
7^3	343	
	-343	
	000	

We then subtract the घन of 2 from 19... the remainder is 11.... step. 3.

We put the next figwre 6 and the next आद्यांक for operation is 116.... step 4.

We then multiply the square of the cube root 2 by 3; the result is 2^2X3 = 12.... step. 5.

We divide the number 116 by 12; and the result is 7; we mulfiply 12X7 = 84 and subtract 84 from 116, the remainder is 32 which together with the next आद्यांक 8 becomes 328. We put this 7 against 2; the whole number is 27.... step 6.

We then make the square of 7, multiply it by the first घनमूल viz.2 and by 3. The number is 294 which is to be subtracted from 328 giving out (328-294=) 34..... step 8.

We then take 3 from above; and write it again 34, giving out the number 343. Subtract from this the cube of 7; this gives the remainder as zero. The process is complete.... step 9.

The फल is 27 which is the घनमूल of the given number, 19683.

It should be remembered that when we reach the stage of घनस्थान in the process of finding out the cube-root, we have to subtract the cube of the number in the पंक्ति; in all other stages, simple division is to be resorted to.

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Apendix : A

Glossary of Technical Terms

(The numbers in the brackets refer to the verses in the book)

अ

अङ्गुल (5) = a kind of a measure of length.
अघन (-स्थान, 28) = non-cube place.
अन्तर (13) sbtraction
अन्त्य (12) = जलधि X 10 = 10^{15}; (15) last, final.
अन्वित (16) = added
अपवर्त्य (18) abbreviating by dividing the number by a common factor.
अब्ज (11) = अर्बुद X 10 = 10^9
अभीष्ट (16) = any desired number.
अयुत (11) = सहस्र X 10 = 10^4
अर्बुद (11) = कोटि X 10 = 10^8
अभिहति (20) = multiplication

आ

आढक (8) = a measure of volume.
आदि (24) = first (from the right)
आद्य (22) = next; (28) = first from right.
आप्त (22) = quotient.
आप्तवर्ग (22) = square of the quotient.

इ

इष्ट (16) = any desirable number.

उ

उत्क्रम (13) = in reverse order; from left to right.
उपान्त्य (15) = pen-ultimate.
उपेत (14) = added
उत्सार्य (19) = moving to the higher or lower rank.

ऊ

ऊन (16) = subtracted, less than.

ए

एक (11) = the एकंस्थान; digital place, first place from the right.
= 1 = 10^0

ऐ

ऐक्य (20) = addition

क

कर (6) = measure of length.
करणसूत्र (13, 15, 18, 19, 22, 24, 28) = Rule for mathematical operation.
कर्ष (4) = a kind of a weight.
काकिणी (2) = a kind of a coin.
कुडव (8) = a measure of volume.
कृति (19) = square.
कोटि (11) = प्रयुत X 10 = 10^7
क्रम (13) = in direct order from right to left.
क्रोश (5) = a measure of length.
क्षेत्र (6) = an area on/in a plane.

ख

खर्व (11) = अब्ज X 10 = 10^{10}
खारी (7) = a measure of volume.

ग

गद्याणक (3) = a kind of a weight.
गुञ्जा (3) = a kind of a weight.
गुण (16) = multiplier, multiplication
गुणक (15) = multiplier
गुणित (16) = multiplication, multiplied.
गुण्य (15) = multiplicand

घ

घन (24) = cube
घनमूल (29) = cube-root

घनस्थान (28) = the cube-place.
घनहस्त (7) = volume measuring a cubic हस्त; a measure of volume.

छ

छिन्न (17) = divided.

ज

जलधि (12) = शंकु X 10 = 10^{14}

ट

टङ्क (9) = a measure of volume
त्यक्त्वा (22) = After subtracting.
√ त्यज् (29) = to subtract.
त्रिघात (24) = multiplication three times.
त्रिघ्न; त्रिनिघ्न (24) multiplied by 3.

द

दण्ड (5) = a measure of length.
दल (22) = one half of the numbers in the column of पंक्ति
दश (11) = 1X10 = 10^1
दशगुणोत्तर (12) = each succeeding ten times (the preceding one)
द्रम्म (2) = a kind of a coin.
द्रोण (8) = measure of volume.
द्विगुण (denom. द्विगुणयेत्, 22) = multiply by 2.
द्विघात (19) = multiplication two times.
द्विघ्न or द्विनिघ्न (20, 22) = multiplied by 2.

ध

धटक (3) = a kind of a weight.
धटिका (9A) = a kind of a measure of volume.
धरण (3) = a kind of a weight.

न

निखर्व (11) = खर्व X 10 = 10^{11}
निघ्न (19) = multiplied.
निवर्तन (6) = an area of 20 *vaṁśa* X 20 *vaṁśa*
निष्क (2) = a kind of a coin

प

पङ्क्ति (22) = a column called पंक्ति
पण (2) = a kind of a coin
पद (22) = result, answer
पल (4) = a kind of a weight
परार्ध (12) = मध्य X 10 = 10^{17}
पाटी (1) = method or the arithmetic.
प्रयुत (11) - लक्ष X 10 = 10^{6}
प्रस्थ (8) = a measure of volume.

फ

फल (16) = result, answer.

भ

भक्त (16) = divided
भज् (with or without सम्, वि) (28) = to divide.
भागहार (18) = operation of division.
भाज्य (18) = the number to be divided, dividend
भुज (= हस्त, 5, 6) side, a measure of length.

म

मण (9, 9A) = a measure of volume.
महापद्म (11) = निखर्व X 10 = 10^{12}
मागधखारिका (7) = a kind of measure of volume.
माष (4) = a kind of a weight.
मूल (22) = square root, cube-root.

य

यव (3) = a kind of a weight and (5) a kind of a measure of length.
युक् (20) = addition.
युक्त (16) = added
युक्ति (14) = addition.
योजन (6) = measure of lenght / distance.

र

राशि (19) = the number, result, answer, expression.
रूपविभाग (16) = division of a number on the basic of its form.

ल

लक्ष (11) = अयुत X 10 = 10^5
लब्ध (22) = quotient.
लब्धकृति (22) = square of the quotient.
लब्धि (16) - quotient.

व

वंश (6) = a measure of length.
वध (20) = multiplication.
वराटक (2) = a kind of a coin.
वर्ग (21) = square.
वर्गमूल (22) = square root
वर्जित (16) = subtracted.
वल्ल (3) = a kind of a weight.
वियुत (14) = subtracted.
विषम (22) = odd.
व्यवकलन (13) = operation of subtraction.
व्यवकलित

श

शंकु (11) = महापद्म X 10 = 10^{13}
शत (11) = दश X 10 = 10^2
शुध् (18) = divide to get a quotient.
शेर (9,9A) = a measure of volume.

स

संकलन / संकलित (13) = operation of addition.
संगुणित (15) = multiplied.
सद्गणित (1) = pure mathematics of real numbers.
सम (18) = common; (22) even.
समत्रिघात (24) = multiplication of the same number thrice.
समद्विघात (19) = multiplication of a number by itself.
समन्वित (20) = added;
समेत (16) = added.
सहस्र (11) = शत X 10 = 10^3
सुवर्ण (4) = a kind of a weight.
स्थानविभाग (17) = division of a number on the basis of places or ranks or levels.

ह

√ हन् (15) = to multiply
हर (18) = quotient
हस्त (5) = a measure of length.
हार (18) = divisor.
हीन (21) = subtraction.
√ हृ = to divide.
हृत (22) = divided.

•••

Appendix B

Modern symbols for the eight mathematical operations.

A. 1. संकलित
2. संकलन
3. योग/संयोग
4. युक्ति
5. उपेत/समन्वित/अन्वित
6. मीलित/सम्मीलित
7. चय, अधिक, अनुबन्ध
8 ऐक्य

= addition is indicated by the plus sign like +, between the two numbers; thus 5+4; it is read as 5 plus 4

B. 1. व्यवकलित
2. व्यवकलन
3. वियोग, वियुत
4. हीन/अपचय
5. विवर्जित
6. ऊन/ऋण

= subtraction is indicated by the minus sign like — between the two numbers; thus 5-4; it is read as 5 minus 4.

C. 1. गुणन
2. हनन/निघ्न
3. हति/अभिहति
4. वध/घात
5. निहति

= multiplication is indicated by the sign X or a dot between the two number; thus 5X4 or 5.4; it is read as 5 multiplied by or into 4.

D. 1. भागहार
2. हृति
3. हार/हर
4. विभक्ति/भक्ति
5. विभजन/विभाजन
6. तक्षण
7. च्छिद्

= division is indicated by the sign ÷ between the two numbers; thus 5 ÷ 4; it is read as 5 divided by 4.

E. 1. वर्ग 2. कृति 3. समद्विघात/द्विघात	square of the number is indicated by the number of the power to which the former is raised; it is written on the right top; thus 5^2; it is read as 5 raised to the power of 2, or 5 squared. The figure at the right top of the number is called as power or index or exponent.
F. 1. वर्गमूल 2. कृतिमूल 3. मूल 4. वर्गपद 5. पद 6. करणी	= square-root of the number is indicated by the sign 2 to the left top and before the number; thus $^2\sqrt{5}$; it is read as root 5 or square root 5. Generally in square roots, the figure 2 is dropped and only one sign is used; thus $\sqrt{5}$
G. 1. घन 2. समत्रिघात / त्रिघात	= cube of a number is indicated by writing the exponent 3 at the right top of the number; thus, 5^3; it is read as 5 cubed or 5 raised to the power of 3.
H. 1. घनमूल 2. घनपद	cube-root of a number is indicated by the sign 3 to the left top and before the number; thus $^3\sqrt{5}$; it is read as cube-root 5
I. सम/समान	= equality of two numbers is indicated by the sign = (two horizontal bars) between the two number; thus, 5 = 5 or X = 5 etc; it is read as 5 is equal to 5 or X is equal to 5.
J. अपवर्तन	= Abbreviation of a fraction by division by a common factor e.g. 10/4 = 5/2 etc.

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Appendix : C

Certain undefined / unstated / unexplained devices / techniques

Certain devices or techniqnes in mathematics, especially is Sanskrit mathematics are not defined or explained or stated any where either in *Līlāvatī* or later mathematical works. Such techniqnes, therefore, require to be learnt from the teacher itself while studying. The following are some of the most simple and basic techniques which are not explained by Bhāskarācārya while explaining the eight mathematical operations. These techniques are, therefore, collected together below in order to avoid confusion :-

1. अंकानां वामतो गतिः :- Though this phrase is very well-known and oft-stated and quoted in Sanskrit mathematics or in dealing with numbers outside the field of mathematics, it is nowhere explained or defined; not only this, but even its source is not traceable in any of the Sanskrit works, mathematical or non-mathematical.

The phrase in its broken form looks like a pāda of a verse in अनुष्टुभ् - metre. And the whole verse is as follows[5]:-

अंकेषु शून्यविन्यासाद्
वृद्धिः स्यात्तु दशाधिका ।
तस्माज् ज्ञेया विशेषेण
अंकानां वामतो गतिः ॥

The whole verse, as it looks and its source indicates, seems to be an example of humour and seems to form part of a stock of phrases in the language, which are meant to be quoted as short, pithy phrases to create amusement and humour in the group of friends. Although it is from outside the mathematical context and thus has no source which is mathematically authentic, it serves our purpose here.

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5. cf. समयोचितपद्यरत्नमाला; I am extremely thankful to Dr. R. P. Goswami, Assistant Librarian, CASS, University of Poona, Pune for tracing the verse to the above-said text.

What the phrase अंकानां वामतो गतिः means is that the order of the number-words written or spoken should be reversed while transforming them into number-symbols. We know that the number could be spoken only in one way, that is, by using the words. But they can be written in two ways : first, we can write the words for the numbers, and/or secondly, we can write the symbols for the numbers. Thus, the word *dvi-sapta* can be spoken only in words; but while put to writing, it can be written as द्विसप्त (with words) or as 72 (with symbols for द्वि and सप्त). It is at the time of writing the number - words in terms of number - symbols that the rule अंकानां वामतो गतिः has to be applied. Thus, while writing the symbols 2 (for द्वि) and 7 (for सप्त), the order, which following the wording will be 27, should be reversed to 72 and will mean the number 'seventy-two' and not 'twenty-seven.' The rule अंकानां वामतो गतिः, therefore, is to be followed at the stage of writing the spoken number-word in terms of a number-symbol.[6]

This technique can be learnt only through the teacher.

2. Numbers 'in hand' :-

While going through a mathematical operation, one faces a situation in which the result contains two, instead of one, numbers with different ranks. In such cases, we take the number of the higher rank as one 'in hand' and carry it on to the next rank while performing the next operation. In such cases we say 'the number is in hand' or 'is carried on further to the next rank.' Consider for exmple the following instance of the addition :-

```
        79
      + 97
   ----------
         1   number in hand
   ----------
योग : =  176
```

The total of the एकंस्थान gives us 16 in which 1 is the number of the higher rank of दश; we carry it further by saying 'it is in hand.'

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6. Why ? for a detailed discussion, cf. M. D. PANDIT, *ibid*, pp. 80-90.

This technique also is nowhere stated but has to be learnt only from the teacher.

3. Rules for knowing whether a given number is divisible by 2, 3 and/or 5 :-

We sometimes want to divide a number by the numbers 2, 3 and/or 5. It is very convenient to know before-hand whether the number is divisible by the given number. The difficulty is all the more felt in the case of very big numbers. In such cases, following rules should be followed before we actually go for the operation of division:-

3.1. Rule for divisibility by 2:

If the given number, however big or small, has the number 2, 4, 6, 8 and 0 at its digital place i.e. एकंस्थान, the number is safely divisible by 2. Thus, to take the example of a big number like the following,

1 2 3 4 5 6 7 8 9 0

Since the number ends in 0, it is definitely divisible by 2; and the division is :

617283945.

One can try by puting 2, 4, 6 and 8 in the एकंस्थान.

3.2. Rule for divisibility by 3 :-

If the total of all the digits in the given number, big or small, is divisible by 3, then the whole number is definitely divisible by 3. Thus, since the total of all the digits in the above - given big number is 45, (i.e. 1+2+3+4+5+6+7+8+9 = 45), the whole number is definitely divisible by 3; and the answer of the division is 411522630.

3.3 Rule for divisibility by 5 :-

If the given number, big or small, has 5 or 0 at its digital place i.e. एकंस्थान, the number can safely be divided by 5. Thus, since the number 1234567890 has 0 at the digit - place, it can be divided by 5; and the result is 246913578 .

By divisibility we mean that after the operation of division is

complete, the remainder is zero.

These rules are also not given in any of the Sanskrit texts on mathematics; they have to be learnt from the teacher at the time of studying the subject.

●●●

Appendix : D

- Word - Numbers -

Besides the number - words एक, द्वि etc. used for the number-symbols, the ancient Indian mathematicians used another type of words for numbers which we may call as 'word-numbers' or 'word-numerals[7].' The mode of signifying numbers by word-numerals is as follows. We find that certain things exist in groups which have a fixed number of elements or entities or constituents. The eyes, for example, are always seen to exist in twos; there is no creature on the earth, or at least a human being ever born, with eyes having with less or more number than two. The Sun or the Moon is always one. The ancient Indian mathematicians collected all such groups, or what may be called in mathematics as sets, with always a fixed number of elements in them and used the words for the sets themselves for signifying the actual mathematical numbers. Thus we have in the verse no 17 the word दिवाकर गुण; दिवाकर = the Sun; besides signifying the number 1, it also signifies the number 12 which can astronomically be interpreted as referring to the 12 forms of the Sun which he assumes in his course through the 12 zodiac signs and hence 12 months. The word द्वि-अंक-इन्दु in verse no. 9A refers to the number 192; द्वि = 2; अंक = 9 and इन्दु = 1. We find such phrases and compounds of word - numberals spread abundantly throughout Sanskrit literature, not only mathematical and astronomical but the non-mathematical also.

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7. for a detailed discussion on this topic and on the different modes of expressing numbers, cf. G. H. Khare, *Saṁśodhakācā Mitra* (in Marathi), Bhārat Itihās Saṁśodhan Maṇḍaḷ Pune, 1959; also Śrīdhar Śāmrao Haṇamante, *Saṁketa Kośa,* Solapur, Śake 1885 (1963 A D); cf. also Oza Gaurishankar Hirācand, *Bhāratīya Prācīna Lipimālā,* Munshi Ram Manohar Lal, New Delhi, 3rd Ed. 1959; cf. also, A. K. Bag, *Mathematics in Ancient and Medieval India,* Chaukhambha Orientalia, Varanasi, 1979, pp. 69f.

It should also be noted that while interpreting these word - numbers, the rule अंकानां वामतो गतिः must also be applied; thus, द्वि - अंक - इन्दु, literally transformed into number - symbols will be 2-9-1; it is then reversed and the actual number meant is 192.

It is impossibe to list all the possible word-numbers for the numbers; the following list of word-numbers, however, is appended here as a sample. It is reproduced from Oza's book entitled *Bhāratīya Prācīn Lipimālā.*

0 = शून्य, ख, गगन, आकाश, अंबर, अभ्र, वियत्, व्योम, अंतरिक्ष, नभ, पूर्ण, रंध्र आदि.
1 = आदि, शशि, इंदु, विधु, चंद्र, शीतांशु, शीतरश्मि, सोम, शशांक, सुधांशु, अब्ज, भू, भूमि, क्षिति, धरा, उर्वरा, गो, वसुंधरा, पृथ्वी, क्षमा, धरणी, वसुधा, इला, कु, मही, रूप, पितामह, नायक, तनु आदि.
2 = यम, यमल, अश्विन, नासत्य, दस्र, लोचन, नेत्र, अक्षि, दृष्टि, चक्षु, नयन, ईक्षण, पक्ष, बाहु, कर, कर्ण, कुच, ओष्ठ, गुल्फ, जानु, जंघा, द्वय, द्वंद्व, युगल, युग्म, नयन, कुटुंब, रविचंद्रौ आदि.
3 = राम, गुण, त्रिगुण, लोक, त्रिजगत्, भुवन, काल, त्रिकाल, त्रिगत, त्रिनेत्र, सहोदरः, अग्नि, वह्नि, पावक, वैश्वानर, दहन, तपन, हुताशन, ज्वलन, शिखिन्, कृशानु, होतृ आदि.
4 = वेद, श्रुति, समुद्र, सागर, अब्धि, जलधि, उदधि, जलनिधि, अंबुधि, केंद्र, वर्ण, आश्रम, युग, तुर्य, कृत, अय, आय, दिश् (दिशा) बंधु, कोष्ठ, वर्ण आदि.
5 = बाण, शर, सायक, इषु, भूत, पर्व, प्राण, पांडव, अर्थ, विषय, महाभूत, तत्व, इंद्रिय, रत्न आदि.
6 = रस, अंग, काय, ऋतु, मासार्ध, दर्शन, राग, अरि, शास्त्र, तर्क, कारक आदि.
7 = नग, अग, भूभृत्, पर्वत, शैल, अद्रि, गिरि, ऋषि, मुनि, अत्रि, वार, स्वर, धातु, अश्व, तुरग, बाजि, छंद, धी, कलत्र, आदि.
8 = वसु, अहि, नाग, गज, दंतिन्, दिग्गज, हस्तिन्, मातंग, कुंजर, द्विप, सर्प, तक्ष, सिद्धि, भूति, अनुष्टुभ्, मंगल आदि.
9 = अंक, नंद, निधि, ग्रह, रंध्र, छिद्र, द्वार, गो, पवन आदि.
10 = दिश्, दिशा, आशा, अंगुलि, पंक्ति, ककुभ्, रावणशिरस्, अवतार, कर्मन् आदि.
11 = रुद्र, ईश्वर, हर, ईश, भव, भर्ग, शूलिन्, महादेव, अक्षौहिणी आदि.
12 = रवि, सूर्य, अर्क, मार्तंड, द्युमणि, भानु, आदित्य, दिवाकर, मास, राशि, व्यय, आदि.
13 = विश्वेदेवाः, काम, अतिगती, अघोष आदि.
14 = मनु, विद्या, इंद्र, शक्र, लोक आदि.
15 = तिथी, घस्र, दिन, अहन्, पक्ष आदि.
16 = नृप, भूप, भूपति, अष्टि, कला आदि.

17 = अत्यष्टि 18. = धृति 19. = अतिधृति 20 = नख, कृति.
21 = उत्कृति, प्रकृति, स्वर्ग. 22 = कृती, जाति. 23 = विकृति.
24 = गायत्री, जिन, अर्हत, सिद्ध आदि. 25 = तत्व. 27 = नक्षत्र, उडु, भ आदि.
32 = दंत, रद, द्विज, दशन आदि. 33 = देव, भ्रमर, त्रिदश, सुर आदि.
40 = नरक. 48 = जगती. 49 = तान.

Word-numbers for fractions :-

1/2 = दल, अर्ध.
1/4 = चरण, पाद, अङ्घि
3/4 = पादत्रय, व्यङ्घि.
1/10 = दशमलव.
1/9 = नवमलव.
1/63 = त्रिषष्टिभाग.

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Appendix : E

॥ अथ ब्रह्मयज्ञः ॥

The following is the text of the ब्रह्मयज्ञ with proper reference to the books from which the text is prepared :-

ॐ अग्निमीळे पुरोहितम् ॥ ऋग्वेद I. I. I.
अग्निर्वै देवानामवमो विष्णुः परमः ॥ ऐतरेय ब्राह्मण I.I.I.
अथ महाव्रतम् ॥ ऐतरेय आरण्यक, 2.I.I.
अथातः संहिताया उपनिषत् ॥ ऐतरेयआरण्यक, 3.I.I.
विदा मघवन् विदा ॥ ऐतरेय आरण्यक, 4..I.I.
महाव्रतस्य पञ्चविंशतिं सामिधेन्यः ॥ ऐतरेय आरण्यक, 4.I.I.
इषे त्वोर्जे त्वा ॥ शुक्लयजुर्वेद वाजसनेयि संहिता, I.I
अग्न आ याहि वीतये ॥ सामवेद I.I.I
शं नो देवीरभिष्टये ॥ अथर्ववेद (पैप्पलाद), I.I.I
अथैतस्य समाम्नायस्य । आश्वलायनश्रौतसूत्र I.I.I
समाम्नायः समाम्नातः ॥ निरुक्त, I.I.
मयरसतजभनलगसम्मितम् ॥ पिंगलछन्दःसूत्र, I.I.
गौः । ग्मा ॥ निघण्टु, I.I.
पञ्चसंवत्सरमयम् ॥ वेदांगज्योतिष, 1
अथ शिक्षां प्रवक्ष्यामि ॥ पाणिनीय शिक्षा, 1
वृद्धिरादैच् ॥ पाणिनीय अष्टाध्यायी, I.I.I.
योगीश्वरं याज्ञवल्क्यम् ॥ याज्ञवल्क्य स्मृति, I.I.
नारायणं नमस्कृत्य ॥ महाभारत, I.I.I.
अथातो धर्मं व्याख्यास्यामः ॥ हारितधर्मसूत्र1, quoted by मन्वर्थमुक्तावलि (२.१)
अथातो धर्मजिज्ञासा ॥ मीमांसासूत्र, I.I.I.
अथातो ब्रह्मजिज्ञासा ॥ ब्रह्मसूत्र, 1.
॥ ॐ शांतिः शांतिः शांतिः ॥

If we examine the sources of the above quotations taken from the different texts, we find there are in all 18 texts which are referred to. They are (in the order of the ब्रह्मयज्ञ):

1. ऋग्वेद
2. ऐतरेय ब्राह्मण
3. ऐतरेय आरण्यक
4. शुक्ल यजुर्वेद or वाजसनेयि संहिता
5. सामवेद
6. अथर्ववेद
7. आश्वलायनश्रौतसूत्र
8. निरुक्त
9. पिंगलछन्दःसूत्र
10. निघण्टु
11. वेदांग ज्योतिष
12. पाणिनीय शिक्षा
13. पाणिनीय अष्टाध्यायी
14. याज्ञवल्क्यस्मृती
15. महाभारत
16. हारितधर्मसूत्र
17. मीमांसासूत्र
18. ब्रह्मसूत्र

Out of these, maximum i. e. 5 references are taken from ऐतरेय आरण्यक alone; the ऐतरेय आरण्यक, therefore, can be taken to represent the आरण्यक literature. The four references from the four main vedic samhitas viz. ऋग्वेद, शुक्लयजुर्वेद, सामवेद and अथर्ववेद represent the Samhita-texts in general. The ऐतरेय ब्राह्मण stands for the ब्राह्मण literature, the आश्वलायन श्रौतसूत्र stands for the whole श्रौत literature. The हारितधर्मसूत्र symbolises the धर्मसूत्र literature. The याज्ञवल्क्यस्मृति stands for स्मृति literature. The महाभारत represents the इतिहास literature. The सूत्रs from the पूर्व मीमांसा and ब्रह्मसूत्र or the उत्तरमीमांसा stand for the यज्ञ philosophy and the उपनिषत् philosophy respectively. The निरुक्त and निघण्टु, the पाणिनीय शिक्षा, the अष्टाध्यायी, the पिंगलछन्दःसूत्र and the वेदांग ज्योतिष represent respectively the five वेदांगs, viz. निरुक्त, शिक्षा, व्याकरण, छन्दस् and ज्योतिष which are accepted as helping the Vedic interpretation. The कल्पसूत्रs, which are also one of the वेदांगs are not included in the above list, though one can include them under the श्रौतसूत्र represented by आश्वलायनश्रौत्रसूत्र.

The list given in the ब्रह्मयज्ञ, therefore, practically covers the whole of Vedic literature, as also the Vedangas.

The Science of pure mathematics, however, is conspicuously absent from the above enumeration; one may include it under

वेदांगज्योतिष, though it must be noted, वेदांग ज्योतिष contains very less of applied mathematics and does not contain pure mathematics at all.

Another important point which deserves notice is if the initial stanzas are taken to represent the type of literature from which they are quoted, the ऐतरेय ब्राह्मण, the ऐतरेय आरण्यक and the आश्वलायन श्रौतसूत्र will represent respectively the ब्राह्मण, आरण्यक and श्रौतसूत्र.

It will be seen that the above ब्रह्मयज्ञ belongs to the Rgvedic recension.

The different Vedas have different ब्रह्मयज्ञ texts. Every Veda starts the ब्रह्मयज्ञ with its own recension. Thus, the ऋग्वेद starts with अग्निमीळे पुरोहितम् । The शुक्लयजुर्वेद begins with इषे त्वोर्जेत्वा. The सामवेद recites first its own अग्न आयाहि वीतये and the अथर्ववेद ब्रह्मयज्ञ commences with the first मंत्र of itself, viz. शं नो देवीरभिष्टये. After the recitation of the initial vedic मन्त्र follow their auxiliary sciences, that is, their वेदांगs.

It is to be specially noted that none of the four ब्रह्मयज्ञ texts counts गणित as the वेदांग.इ*

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Appendix F

English eqnivalents for place - values of the place - names given by Bhāskarācārya.

एकम् - digital place; digit = numbers from 0 - 9.

दश - decimal place; ten = 10^1

शतम् - hundred's place; hundred = 10^2 (2nd power of 10)

सहस्र - thousand's place; thousand = 10^3 (3rd power of 10)

प्रयुत - million's place; million; a thousand thousand = 10^6 (6th power of 10)

महापद्म - a million million's place; a million million, billion; second power of a million; $(\text{million})^2$

The English language has no separate names or words for the other terms given by Bhāskarācārya. It, however, contains two more terms for which there are no equivalents in Sanskrit; they are :

Eng. *trillion* = a million million million's place; a million million million; third power of a million i.e. $(\text{million})^3$; this number and place are equivalents to Bhāskarācārya's परार्ध x 10 = 18^{th} power of 10 i.e. 10^{18}.

Eng. *decillion* = 10^{th} power of million = $(\text{million})^{10}$ which comes to the 60^{th} power of 10, i.e. 10^{60} (1 with 60 zeroes). It has no Sanskrit equivalent; *decem + million = decillion* Etymologically, Eng. *billion = bi + million* and *trillion = tri + million.* In France and US, billion = a thousand million i.e. = 10^9 which is Bhāskarācārya's अब्ज. In Britain, the value is 10^{12}. Incidentally, Bhāskarācārya's परार्ध is equal to half-lfte, that is, 50 divine years of ब्रह्मदेव. परार्धम् = परस्य अर्धम् = half of the highest.

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Editions, Translations, Commentaries and Bibliography for *Līlāvatī.*

Sr. No.	Author	Commentary	Śaka	A.D.
१	रामकृष्ण	गणितामृतलहरी	१२६१	१३३९
२	नृसिंह	वासनावार्तिक	१२७२	१३५०
३	नारायण भट्ट	गणितकौमुदी	१२७८	१३५६
४	गंगाधर	गणितामृतसागरी	१३४२	१४२०
५	लक्ष्मीदास	गणितचिंतामणी	१४२२	१५००
६	रामकृष्ण देव	मनोरंजना		15th cent.
७	सूर्यदास	गणितामृतकुपिका	१४६०	१५३८
८	गणेश दैवज्ञ	बुद्धिविलासिनी	१४६७	१५४५
९	फैझी	भाषांतर	१५०९	१५८७
१०	मुनीश्वर	नि:सृष्टिदूति (मरीचिका)	१५३०	१६०८
११	रंगनाथ	वासनाभाष्य (मितभाषिणी)	१५४२	१६२०
१२	महीधर	लीलावतीविवरण	१५५७	१६३५
१३.	महीदास	--	१५०९	१५८७
१४	जेम्स टेलर	भाषांतर (इंग्रजी)	-	१८१६
१५	एच्. टी. कोलब्रुक	'' (इंग्रजी)	-	१८१७
१६	एरंडोलकर	'' (मराठी)	-	१८८९
१७	खानापूरकर शास्त्री	'' (मराठी)	-	१८९७
१८	रावेरकर शास्त्री	प्रकाशिका (हिंदी)	-	१८९३
१९	राजे शिवप्रसाद	'' (हिंदी)	-	१८७६
२०	पंडित जियाराम	'' (हिंदी)	-	१८९३
२१	पंडित दयानाथ	'' (हिंदी)	-	१९५९
२२	पंडित सुधाकर द्विवेदी	चंद्रप्रभा (हिंदी)	-	१९१०
२३	पंडित रामस्वरूप शर्मा	स्वरूपप्रकाश (हिंदी)	-	१९१५
२४	हरिणचंद्र बानर्जी	भाषांतर (इंग्रजी)	-	१९३०

Besides, कृष्ण दामोदर, परशुराम, वृंदावन, धनेश्वर, पंडित जीवानंद विद्यासागर (1900AD) have also written their commentaries on Bhāskarācārya's *Līlāvatī.*

25) M. D. PANDIT, *Mathematics As Known to the Vedas,* Vol. I, Indian Books Centre, New Delhi, 1992.

26) D. E. SMITH, *History of Mathematics,* Vols. I & II, Dover Publications Inc., New York, 1923.

27) ना. ह. फडके, लीलावती पुनर्दर्शन (in Marathi), प्रजापति भुवन, गोखले रोड, दादर, मुंबई, 1971.

The list of commentaries on *Līlāvatī* is not exhaustive; not only this, but even some of the commentaries are incomplete, unintelligible and currupt. There may be some more, scattered here and there, which are still not brought to light; for details, cf. शं. बा. दीक्षित, भारतीय ज्योतिषशास्त्र, Aryabhūṣaṇa Press, Pune, 1931, pp. 246-254, 257, 279 etc.

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ERRATA

incorrect	correct
widependent (foreword, p.3. line 15)	independent
whanimously (foreword, p3. line 16)	unanimously
indics (p.36, line 9)	indices
p.51	

$$\begin{array}{l} 420 \\ \underline{12} \\ 1620 \end{array} \qquad \begin{array}{r} 420 \\ \underline{\quad 12} \\ 1620 \end{array}$$

Other publications by
M. D. PANDIT.

1. **A Concordance of Vedic Compounds Interpreted by Veda,** Centre of Advanced Study in Sanskrit, University of Poona, Pune, 1989.

2. **प्राचीन भारतीय जलशास्त्र** (Ancient Indian Hydrology), संस्कृत प्रगत अध्ययन केन्द्र, पुणे विद्यापीठ, पुणे, १९९०.

3. **Zero in Pāṇini,** Centre of Advanced study in Sanskrit, University of Poona, Pune, 1990.

4. **The Ṛgvedic Family-Maṇḍalas - A Statistical study,** Centre of Advanced study in Sanskrit, University of Poona, Pune, 1991.

5. **Mathematics As Known to the Vedic Saṁhitās,** Indian Books Centre, 40/5, Shakti Nagar, New Delhi, 1992.